Announcements

- See webpage for handouts / notes

- HW due next Wed. at start of class
Peasant multiplication

```
PEASAN M U L T I P L Y (x, y):
    prod ← 0
    while x > 0
        if x is odd
            prod ← prod + y
        x ← ⌊x/2⌋
        y ← y + y
    return p
```

Ex:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>prod</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>456</td>
<td>0</td>
</tr>
<tr>
<td>561</td>
<td>912</td>
<td>456</td>
</tr>
<tr>
<td>130</td>
<td>1824</td>
<td>1368</td>
</tr>
<tr>
<td>15</td>
<td>3648</td>
<td>-</td>
</tr>
<tr>
<td>-7</td>
<td>7296</td>
<td>5016</td>
</tr>
<tr>
<td>-3</td>
<td>14592</td>
<td>12312</td>
</tr>
<tr>
<td>1</td>
<td>29184</td>
<td>26904</td>
</tr>
<tr>
<td>0</td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>

- earliest written record is 1650 BC
  (it was already old!)

- still taught in Europe in late 1900's.
Why correct? (This is non-trivial!)

Key:

\[ x \cdot y = \begin{cases} 0 & \text{if } x = 0 \\ \left\lfloor \frac{x}{2} \right\rfloor (y+y) & \text{if } x \text{ is even} \\ \left\lfloor \frac{x}{2} \right\rfloor (y+y) + y & \text{if } x \text{ is odd} \end{cases} \]
Harder: US constitution

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers.... The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative. ...

But how?

Today: Huntington-Hill method
- proposed in 1911
- adopted in 1941

(50 states, 435 representatives)
Algorithm:

\[ \text{APPORTIONCONGRESS}(\text{Pop}[1..n], R): \]

\[ PQ \leftarrow \text{NEWPRIORITYQUEUE} \]

\[ \text{for } i \leftarrow 1 \text{ to } n \]

\[ \text{Rep}[i] \leftarrow 1 \]

\[ \text{Insert}(PQ, i, \text{Pop}[i]/\sqrt{2}) \]

\[ R \leftarrow R - 1 \]

\[ \text{while } R > 0 \]

\[ s \leftarrow \text{EXTRACTMAX}(PQ) \]

\[ \text{Rep}[s] \leftarrow \text{Rep}[s] + 1 \]

\[ \text{Insert}(PQ, s, \text{Pop}[s]/\sqrt{\text{Rep}[s](\text{Rep}[s]+1)}) \]

\[ R \leftarrow R - 1 \]

\[ \text{return } \text{Rep}[1..n] \]
A **bad example:**

**BECOME A MILLIONAIRE AND NEVER PAY TAXES:**
Get a million dollars.
Don’t pay taxes.
If you get caught,
Say “I forgot.”

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**specific instructions!**

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Why?
In this class:

3 parts to every algorithm:

1. pseudo code

2. runtime analysis - (big-o)
   (mostly)

3. proof of correctness

+ (sometimes) 4. space
This week: why you should have paid attention in 135.

Topics: - pigeonhole principle
        - counting - combinations, permutations, ...
        - probability
        - proofs
        - Logic
        - graphs
        - sets
        - big O
        - recursion + induction
Runtime:

What is big-O?

worst case running time

mathematical notion of upper bounding

Why use it?
Formal definition:

Let $f$ and $g$ be functions from $\mathbb{R} \rightarrow \mathbb{R}$ (or $\mathbb{Z} \rightarrow \mathbb{R}$). We say that:

$$f(n) = \theta(g(n))$$

if there exist constants $C > 0$ such that

$$|f(n)| \leq C |g(n)|$$

for all $n > n_0$. 
Ex: \( f(x) = x^2 + 2x + 1 \) is \( \Theta(x^2) \)

**Proof:** Need to find \( C \) and \( n_0 \)

\[
x^2 + 2x + 1 \leq \frac{x^2 + 2x^2 + x^2}{1} \quad (\text{if } x \to 0)
\]

Let \( C = 4 \) and \( n_0 = 1 \)

Then \( x^2 + 2x + 1 \leq 4x^2 \)
**Thm:** Let \( f(x) \) be a polynomial. So \( f(x) = \sum_{i=0}^{h} a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) where \( a_0, a_1, \ldots, a_n \in \mathbb{R} \). Then \( f(x) = O(x^n) \).

**Central idea of proof:** What \( c \) to use?

\[
C = |a_n| + |a_{n-1}| + \ldots + |a_0|
\]
Other useful functions:

(remember 180?)

\[ n^n \gg n! \gg 2^n \ldots \]
Induction: Recursion’s twin

A method of proving a statement which depends upon the statement holding at smaller values.

Can think of this as “automating” a proof:

true for $n = 1$

assume true for $< n$

show also true at $n$
Ex: \[ \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2} = \Theta(n^2) \]

**pf:**

**Base case:** \( n = 1 \)

\[ \sum_{i=1}^{1} i = 1 \]

\[ \frac{(n+1)n}{2} = \frac{(1+1)!}{2} = 1 \]

**I.H.:** Assume for \( k < n \), that \( \sum_{i=1}^{k} i = \frac{(k+1)k}{2} \)

**I.S.:** \( \sum_{i=1}^{n} i = \left( 1 + 2 + 3 + 4 + \cdots + (n-1) + n \right) \)

\[ \sum_{i=1}^{n-1} i + n = \frac{(n-1+1)(n-1)}{2} + n \]

\[ = \frac{n^2}{2} + n = \frac{n(n-1)+2n}{2} = \frac{n(n-1+2)}{2} \]

\[ = \frac{n(n+1)}{2} \]