Announcements

- Next hw up tonight due 1 week after Mon/Tues (oral grading)
**NP-Completeness**

A problem is **NP-Complete** if it is both:

- in **NP**
- NP-Hard

More of what we *think* the world looks like.

polynomial hierarchy
Thm (Cook-Levin): (The first known NP-hard problem)
Circuit Satisfiability is NP-complete.

"pf": any turing machine can be turned into a circuit
To prove NP-hardness of A:

Reduce a known NP-hard problem A' to A.
Last time: NP-Hard Problems

- SAT (from circuit-SAT)
- 3SAT (from circuit-SAT)
- Independent Set (from 3SAT)
How? Transform known hard problem to new one!

Ex: \((a \lor b \lor c) \land (b \lor c \lor d) \land (\overline{a} \lor c \lor d) \land (a \lor b \lor d)\)
Another: clique

A clique in a graph is a subgraph which is complete - all possible edges are present.

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} \\
\propto O(n^k)
\]

A graph with maximum clique size 4.
Decision version: Does $G$ have a clique of size $k$?

**NP-Complete:**

1. In NP. Why?
   How to verify a "yes"?
   Given a set of $k$ vertices, it takes $O(nk)$ time to check if all edges are present.
2. What should we reduce to k-clique?

Ind Set $\rightarrow$ k-clique

Given $G$, answer if ind. set of size $k$ is present.

Transform $G$ to $\overline{G}$

$\overline{G}$
So:

graph $G$ $\xrightarrow{O(n)}$ complement graph $\overline{G}$

largest independent set $\xleftarrow{\text{trivial}}$ largest clique

$\text{MaxClique}$
Next: Vertex Cover:
A set of vertices that touches every edge

k-Vertex Cover: Does G have a vertex cover of size k?

In NP:
- Given a vertex set of size k
- Make a list of edges, $O(n^2)$
- Check that each edge has endpoint in the set, $O(\chi(k))$
NP-Hardness: reduce what to this?

Ind set, S

Key: V-S is a vertex cover since no edges "inside" S

Gwen G + K answer: is the ind. set of size k?

Ask: is there a vertex cover of size n-k?
graph $G = (V, E)$ \quad \text{trivial} \quad \text{graph} \ G = (V, E)$

largest independent set $V \setminus S$

$\text{size } k$

$O(n)$

smallest vertex cover $S$

$\text{size } \leq k$
Next: Graph Coloring

A $k$-coloring of a graph $G$ is a map $\phi: V \to \{1, 2, \ldots, k\}$ that assigns one of $k$ "colors" to each vertex so that every edge has two different colors at endpoints.

Goal: use few colors.
2-coloring? polynomial

Decision version: 3-colorable: Can $G$ be 3-colored?

In NP:
Given a coloring, check no edge has same color at both endpoints.
$O(n^2)$
NP-Hard: Reduce 3-SAT.

Given a formula $\Phi$, make $G_\Phi$.

We'll use gadgets, which each incorporate bits of the clause.

1. **Truth gadget**

   ![Diagram]

   will use 3 colors, since all edges present.
(2) Variable gadget:

Also - for each variable, make a △ with vertex \( x \).

(So \( a \) and \( \bar{a} \) are set to T/F or F/T, by coloring.)
3. **Clause gadget:** joins 3 of the variable vertices together to the \( T \) vertex.

A clause gadget for \((a \lor b \lor \overline{c})\).

If all variables are colored false, can't 3-color (some cases...)
$3$-coloring $\iff$ satisfiable:

- If satisfiable, then each clause has "true" variable in valid $3$-coloring.

$\Leftarrow$: if $3$-coloring gives satisfying assignment.

A clause gadget for $(a \lor b \lor \overline{c})$. 

A clause gadget for $(a \lor b \lor \overline{c})$. 
Picture:
So:

3CNF formula \(\rightarrow\) graph

\(O(n)\)

\(\downarrow\)

3COLORABLE

\(\leftarrow\)

TRUE OR FALSE

trivial

TRUE OR FALSE