CS314 - NP-Hardness

Announcements
- Oral grading next Tuesday
- Survey today
**Hamiltonian Cycle**

A cycle in a graph which visits each vertex exactly once.

How to find?

Ham. path
(no Ham cycle)
$(n-1)!$
Note: Not the same as an Eulerian cycle!

Thm: $G$ has an Euler cycle if and only if every vertex of $G$ has even degree.
Q: Does $G$ have a Hamiltonian cycle? (Yes/no decision problem)

In $NP$:

Given an ordering $v_1, \ldots, v_n$
check that it is a cycle.
NP-Hard: Reduce vertex cover to Hamiltonian cycle:

Given a graph $G$ and integer $k$, answer yes/no if $G$ has a vertex cover of size $k$.

(Use Hamiltonian cycle as a subroutine)

More gadgets!

Make a new graph $G'$:
1. For each edge $uv$, create an edge gadget in $G'$ with 12 vertices and 14 edges.

The route through this will correspond to $u \xrightarrow{v} v$, or both $u \xrightarrow{v}$ and $v \xrightarrow{u}$ being in the cover.
Note: Only 3 possible ways for the Hamilton cycle to go through:

\[ u \text{ in cover} \]

\[ v \text{ in cover} \]

\[ u \leftrightarrow v \text{ in cover} \]
① k cover vertices, numbered 1 to k.

③ For each vertex u, string together all the edge gadgets into a vertex chain $e_{G'}$. Then connect chains to cover vertices on either end.
So for a vertex $v$: 

The vertex chain for $v$: all edge gadgets involving $v$ are strung together and joined to the $k$ cover vertices.
Bigger example:
Now:  \[ \Rightarrow \]

If \( E(v_1, v_2, \ldots, v_k) \) is a cover in \( G \), then we can get a Ham. cycle in \( G' \):

Start at \( v_1 \), go through vertex chain \( E(v_1, v_2) \), then go to \( v_3 \), a chain for \( v_2 \) etc.
Header: ∈

Consider any Ham cycle in G'.

Must alternate cover vertices and vertex chains.

Any vertex chain taken will give the k vertices in a cover.

(A bit more work to do this...)
Ham cycle

Vertex cover instance \((G, k)\)

\(O(m+n)\) -> Build \(G'\)

Ham cycle?

yes/no <--- yes/no
Traveling Salesman

Given n cities along with (all) pairwise distances between them, what is the shortest tour of all the cities?

Decision version:
Is there a tour of length at most k?
NP-Hard: Hamilton cycle ↔ TSP

Input: unweighted graph $G$

Construct $G'$: same vertices
- if $e \in G$, put $e \in G'$ with weight 1
- if $e \notin G$, put $e \in G'$ with weight 2

$G \rightarrow G'$

Q: does $G'$ have TSP tour of length $n$?
Subset Sum

Given a set of numbers \( X = \{ x_1, x_2, \ldots, x_n \} \) and a target \( t \), is there a subset \( Y \) of \( X \) summing to \( t \)?

Ex: (see recursion notes!)

\( n \cdot t \)
NP-Hard: Reduction from vertex cover:

Given G and k.

1. Number G’s edges from 0 to m - 1. Put \( b_i := 4^i \) in \( X \) for each edge \( i \).

2. For each vertex \( v_i \), put

\[
|X| = m + n
\]

\[
4^m + 4^i + 4^5 + 4^7
\]

\[
\sqrt{5}
\]
So everything in $X$ is a base-$4$ number:
- $m$th digit is $1$ if it is a vertex
- $i$th digit is $1$ if integer represents edge $i$ or one of its endpoints.

Then set $t = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$