CSCI 3100

Graph NP-Hard problems
Announcements

- No office hours today
- Next HW: more NP-Hardness + oral grading
Lastly, NP-hard problems:
- SAT
- 3SAT
- Independent Set

How? Reductions!

To prove any other problem A is NP-hard, we'll use a reduction:
Reduce a known NP-hard problem to A.
A clique in a graph is a subgraph which is complete — all possible edges are present.

A graph with maximum clique size 4.

How could we check if G has a clique of size k?

Take all size k subgraphs, check if all edge are present bet those vertices:

\( O(k^n) \rightarrow \binom{n}{k} \cdot \frac{k \cdot n}{k} = O(kn^{k+r}) \)
Decision version: Does $G$ have a clique of size $k$?

Input: $G, k$

Output: Yes/No

This is NP-Complete:

1. In NP. Why?
   Given the $k$ vertices in the clique, I can verify all edges are present in $O(nk)$. 
What should we reduce to $k$-clique?

Ind. set: Given $G$ and $k$, are there $k$ vertices with no edges between them?

Given $G$, create $\overline{G}$, the complement of $G$:
- $\overline{G}$ will have the same vertex set as $G$
- $e \in \overline{G} \iff e \notin G$

$G$
So:

G has ind set of size k

\( \iff \) \( \overline{G} \) has \( k \)-clique

if G doesn't have edges b/t k vertices, \( \overline{G} \) will (and vice versa)

Conversion: \( O(n^2) \) time
Next: Vertex Cover:
A set of vertices which touches every edge in $G$.

**k-Vertex cover (decision version):**
Given $G$ and $k$, does $G$ have a cover of size $k$?

In NP:
Given $k$ vertices $O(n^2)$, check in $O(m)$ time that all edges are "covered."
NP-Hardness: reduce what?
(propably clique or nd set!)

Key: If $S$ is independent set, what is $V-S$?

$V-S$ is a vertex cover.
- Edges either go from $S$ to $V-S$, or stay in $V-S$. 

So simple reduction!

Given $G + k$ to independent set, ask if $\exists$ vertex cover of size $n-k$.

$
\Rightarrow \quad$ Suppose ind set of size $k \Rightarrow$ vertex cover of $n-k$

$
\leq \quad$ If cover of size $n-k$, that means no edges b/t any of $k$ vertices not in cover

$\Rightarrow$ ind set of size $k$.

\[
\begin{array}{l}
\text{graph } G = (V, E) \quad \text{trivial} \quad \text{graph } G = (V, E) \\
\text{largest independent set } V \setminus S \quad \text{MinVertexCover} \quad \text{smallest vertex cover } S
\end{array}
\]
Next: Graph Coloring

A \( k \)-coloring of a graph \( G \) is a map: \( c : V \rightarrow \{1, \ldots, k\} \) that assigns one of \( k \) "colors" to each vertex so that every edge has \( k \) different colors at its endpoints.

Goal: Use few colors

Peterson is 3 colorable
Aside: this is famous!
Ever heard of map coloring?

Famous theorem: 4 color theorem
Every planar $G$ is 4-colorable.
Thm: 3-colorability is NP-complete.

(Decision version: Given G, output yes/no)

In NP:

Give you a coloring \( c : V \rightarrow \{1, \ldots, 3\} \),

in \( G \), check no edges

b/t vertices of same color.
NP-Hard:
Reduction from 3SAT.
Given formula for 3SAT $\Phi$, we'll make a graph $G_{\Phi}$.

$\Phi$ will be satisfiable $\iff G_{\Phi}$ can be 3-colored.

Key notion: Build gadgets!

1. Truth gadget - one

Must use 3 colors - establishes a "true" color.
2) Variable gadget - one per SAT variable (n total)

So both $x_i$ and $\overline{x_i}$ can't be true

One of these will be colored true/false, but both $x_i$ and $\overline{x_i}$ can't be true
Clause gadget:

For each clause, join 3 of the variable vertices to the "true" vertex from the truth gadget.

A clause gadget for \((a \lor b \lor \overline{c})\).

Idea: If all inputs are colored False, can't 3-color:

Case analysis
3 coloring of $G_{\phi}$ is satisfiable.

Proof:
A 3-colorable graph derived from the satisfiable 3CNF formula
\((a \lor b \lor c) \land (b \lor \bar{c} \lor \bar{d}) \land (\bar{a} \lor c \lor d) \land (a \lor \bar{b} \lor \bar{d})\)
Time to build $G_\Phi$:

- $3n$ on vertex gadgets
- $O(1)$ on truth gadget
- $O(m)$ to build clause edges

$\# \text{ clauses}$

So:

$O(n+m)$
Next time:
- More reductions
- Plus some non-graph problems!