CSA 3100

More LP
Welcome back!

Today:

- HW: may submit by Friday morning
- Next HW - due next Friday
- Sample final coming next week
- Oral grading next Friday (optional)
- Review Session: last day of class
- Final: Friday at 8am
  4 Cheat Sheets
In a linear program, we are given a set of variables. The goal is to give these real values so that:

1. We satisfy some set of linear equations or inequalities.
2. We maximize or minimize some linear objective function.
An example: Maximize profit

A chocolate shop produces 2 products

- Type 1, worth $1 each
- Type 2, worth $6 each

Constraints:

- Can only produce 200 of type 1 per day
- And at most 300 of type 2
- Total output per day of both is ≤ 400

LP:

maximize: \hspace{1cm} \text{obj} \Rightarrow \text{fun} \hspace{1cm} X_1 + 6X_2

s.t. \hspace{1cm} X_1 \leq 200
           X_2 \leq 300
           X_1 + X_2 \leq 400
           X_1, X_2 \geq 0
LP:

Optimum point: Profit = $1900

Constraints:
- \( x_1 \leq 200 \)
- \( x_1 \geq 0 \)
- \( x_2 \geq 0 \)
- \( x_1 + x_2 \leq 400 \)

Graph showing feasible region and optimal solution point.
Connections to other problems:

It turns out that LPs are powerful enough to express many types of problems.

In a sense, we solve many problems by reducing them to an LP:
Ex: Flows + Cuts

Input: directed $G$ w/ edge capacities $c(e)$ + $s, t \in V$

Goal: Compute flow $f: E \rightarrow \mathbb{R}$

s.t.

1. $0 \leq f(e) \leq c(e)$
2. $\forall v \neq s, t,$
   \[ \sum_{(u \rightarrow v)} f(u \rightarrow v) = \sum_{(v \rightarrow w)} f(v \rightarrow w) \]

Make an LP:

Maximize:

- Flow into $v$
- "Variable": flow $f(e)$ on each edge
- Flow out of $V$

s.t.

- $\forall u \rightarrow v,$ $f(u \rightarrow v) \geq 0$
- $f(u \rightarrow v) \leq c(u \rightarrow v)$
Related: Min cuts \((S, T)\)

Use indicator variables:

\[
S_v = \begin{cases} 
0 & \text{if } v \in T \\
1 & \text{if } v \in S 
\end{cases}
\]

\[
X_{u \to v} = \begin{cases} 
1 & \text{if } u \in S \text{ and } v \in T 
\end{cases}
\]
The LP:

Minimize \( \sum_{u \rightarrow v} C_{u \rightarrow v} X_{u \rightarrow v} \)

s.t. For each cut \( u \rightarrow v \):

\( X_{u \rightarrow v} + S_v - S_u \geq 0 \quad \forall u \rightarrow v \)

\( X_{u \rightarrow v} \geq 0 \quad \forall u, v \)

\( S_s = 1 \)

\( S_t = 0 \)
Note:
For that example, a solution to flow cuts would yield optimal LP solution. The reverse is not obvious! LP might have strange fractional answer which doesn't describe a cut. It can be shown that this won't happen but not obvious...
Duality:

Recall our chocolate:

LP: \[
\begin{align*}
\text{max } & \quad x_1 + 6x_2 \\
\text{s.t. } & \quad x_1 \leq 200 \\
& \quad x_2 \leq 300 \\
& \quad x_1 + x_2 \leq 400 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Can we check that this is best?

\[
\begin{align*}
\text{max} & \quad x_1 + 6x_2 \\
\text{s.t.} & \quad \begin{cases}
    x_1 \leq 200 \\
    x_2 \leq 300 \\
    x_1 + x_2 \leq 400 \\
    x_1, x_2 \geq 0
\end{cases}
\end{align*}
\]

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Play w/ inequalities:

1. $+ 6 \cdot 2$

\[
\begin{align*}
x_1 & \leq 200 \\
6x_2 & \leq 1800
\end{align*}
\]

\[
\Rightarrow x_1 + 1800 = 2000
\]
Interesting!

These 2 inequalities tell us that we couldn’t ever beat $2000.

But recall soln was $1900—can we get a better combo?

\[
\begin{align*}
\text{s.t.:} & \quad x_1 + 6x_2 \\
& x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{align*}
\]

Play: \(0 \cdot \overline{1} + 5 \cdot \overline{2} + 1 \cdot \overline{3}\)

\[
5 \left( x_2 \leq 300 \right) \\
\leq 5x_2 \leq 1500 \\
1 \left( x_1 + x_2 \leq 400 \right)
\]

\(\text{add } x_1 + 6x_2 \leq 1900\)

Find the magic multipliers.
These multipliers are a certificate of optimality.

No valid solution can ever beat $1900.

But how do we find these magic values??

In this, we had three “≤” inequalities.

So, the goal is to find 3 multipliers: $y_1$, $y_2$, and $y_3$. 
### Multiplier vs. Inequality

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 200$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 300$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + x_2 \leq 400$</td>
</tr>
</tbody>
</table>

**Result:**

$\begin{align*}
   y_1 (x_1 \leq 200) \\
   y_2 (x_2 \leq 300) \\
   y_3 (x_1 + x_2 \leq 400) \\
\end{align*}$

$\Rightarrow \sum (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$

**Note:** Make the left side look like the original max/min goal so right will be an upper bound.
So here:

\[(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3\]

Means:

\[x_1 + 60x_2 \leq 200y_1 + 300y_2 + 400y_3\]

If:

\[\begin{align*}
\sum & y_1, y_2, y_3 \geq 0 \\
& y_1 + y_3 \geq 1 \\
& y_2 + y_3 \geq 6
\end{align*}\]

Any \(y_i\)'s would give an upper bound!

We want the best one to minimize another LP!
Duality:

\[ \text{max} \quad 10x_1 + 6x_2 \]
\[ \text{s.t.} \quad x_1 \leq 200 \]
\[ x_2 \leq 300 \]
\[ x_1 + x_2 \leq 400 \]
\[ x_1, x_2 \geq 0 \]

\[ \text{min} \quad 200y_1 + 300y_2 + 400y_3 \]
\[ \text{s.t.} \quad y_1 + y_3 \geq 1 \]
\[ y_2 + y_3 \leq 6 \]
\[ y_1, y_2, y_3 \leq 0 \]

Any solution to bottom is upper bound to top LP.

⇒ If we can find primal/duals that are equal, both are OPT,

\[
\text{primal} \quad (x_1, x_2) = (100, 300)
\]
\[
\text{Dual} \quad (y_1, y_2, y_3) = (0, 5, 1)
\]
This is just like max flow/ min cut duality, in a way.

Works for any LP:

\[ \text{primal } \text{OPT} \rightarrow \text{feasible} \rightarrow \text{primal } \text{OPT} \]

\[ \text{dual } \text{OPT} \rightarrow \text{feasible} \rightarrow \text{dual } \text{OPT} \]

\[ \text{This gap } \rightarrow \text{the duality gap } \rightarrow \text{is } = 0. \]
In general:

**Primal LP**

\[
\begin{align*}
\text{max} \quad & \mathbf{c}^T \mathbf{x} \\
\text{s.t.} \quad & A \mathbf{x} = \mathbf{b} \\
\quad & \mathbf{x} \geq 0
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min} \quad & \mathbf{y}^T \mathbf{b} \\
\text{s.t.} \quad & \mathbf{y}^T \mathbf{A} = \mathbf{c}^T \\
\quad & \mathbf{y} \geq 0
\end{align*}
\]

Recall our chocolate:

\[
\begin{align*}
\text{max} \quad & x_1 + 6x_2 \\
\text{s.t.} \quad & x_1 \leq 200 \\
\quad & x_2 \leq 300 \\
\quad & x_1 + x_2 \leq 400 \\
\quad & x_1, x_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} \quad & 200y_1 + 300y_2 + 400y_3 \\
\text{s.t.} \quad & y_1 + y_2 \geq 1 \\
\quad & y_2 + y_3 \geq 6 \\
\quad & y_1, y_2, y_3 \geq 0
\end{align*}
\]