Generalized Persistent Homology: Part II, Why Homology?

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SLU Topology Seminar
Goals

- Understand the underlying structure of persistent homology
- Use more general collections of topological spaces, not just filtrations
- Do we have to use homology?
### Persistent Homology Recap

#### Filtration

\[ X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow \cdots \]

A sequence of topological spaces with maps (often inclusion) between them.

Note: can be indexed by rationals, reals, ...

#### Homology of Filtration

\[ H_k(X_0) \rightarrow H_k(X_2) \rightarrow \cdots \rightarrow H_k(X_n) \rightarrow \cdots \]

#### Persistent Homology

\[ H^p_k(X_t) = im(H_k(X_t) \rightarrow H_k(X_{t+p})) = im(f_{t, t+p}) \]

where \( f_{\alpha, \beta} : H_k(X_{\alpha}) \rightarrow H_k(X_{\beta}) \) is the map induced by the include \( X_{\alpha} \rightarrow X_{\beta}. \)
Birth

An cycle \( c \in H_k(X_t) \) has birth time \( t \) if \( c \notin \text{im}(H_k(X_s) \to H_k(X_t)) \) for any \( s < t \).

Death

The death time of \( c \) is the smallest \( u \) such that the map \( f_{t,u} : H_k(X_t) \to H_k(X_u) \) maps \( u \) to 0.
Persistence Module

Definition

$\mathcal{P}H_k(X)$ is the submodule of $H_k(X_0) \oplus H_k(X_1) \oplus \cdots \oplus H_k(X_n)$ generated by elements of the form $(0, \ldots, 0, c, f_{\alpha,\alpha+1}(c), \ldots, f_{\alpha,\beta}(c) = 0, \ldots 0)$ where $c \in H_k(X_\alpha)$ has birthtime $\alpha$.

Note: this is equivalent to the original definition (due to Carlsson and Zomordian) of the persistence module as a graded $\mathbb{F}[t]$-module.
Krull-Remak-Schmidt

**Theorem**

If $M$ is a Noetherian Artinian module the $M$ decomposes uniquely into direction summands

$$M \cong M_1 \oplus \cdots \oplus M_n$$

Recall that the standard persistence algorithm calculates birth and death pairs. Each of these pairs is a summand in the decomposition of the persistence module.

$$\mathcal{PH}_k(X) = \bigoplus_i F(b_i, d_i)$$

where $F(b, d) = 0 \oplus \cdots \oplus F \oplus \cdots F \oplus 0 \oplus \cdots \oplus 0$ has non-zero terms for $b \leq t < d$. 
What is it? (Abstract non-sense?)

Provides a formal framework for mathematical objects, their properties, maps between them, ...
What is it? (Abstract non-sense?)

Provides a formal framework for mathematical objects, their properties, maps between them, ...

Examples

- Sets
- Groups, Abelian groups
- Rings, fields, modules, vector spaces
- Topological spaces
- ...
Definition

A category $\mathcal{C}$ has the following:

- $\text{Obj}(\mathcal{C})$, the objects
- $\text{Hom}(\mathcal{C})$, the morphims (or maps) from on object in $\text{Obj}(\mathcal{C})$ to another. If $A, B \in \text{Obj}(\mathcal{C})$ then $\text{Hom}(A, B)$ is the space of all morphisms from $A \to B$.
- A binary operation $\circ$, called composition, of two morphisms. Formally, if $A, B, C \in \text{Obj}(\mathcal{C})$ then $\circ : \text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C)$ that satisfies:
  - Associativity, for $f \in \text{Hom}(A, B), g \in \text{Hom}(B, C), h \in \text{Hom}(C, D)$, $f \circ (g \circ h) = (f \circ g) \circ h$.
  - Identity, for any object $A \in \text{Obj}(\mathcal{C})$, there is a morphism $1_A \in \text{Hom}(A, A)$, the identity morphism, such that for any $f \in \text{Hom}(A, B)$, $1_A \circ f = f = f \circ 1_B$.
## Examples of Categories

<table>
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<tr>
<th>Category</th>
<th>Objects</th>
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<tr>
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<td>Top</td>
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<td>Grp</td>
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<td>Ab</td>
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<td>group homomorphism</td>
</tr>
<tr>
<td>Vect(_k)</td>
<td>vector space over the field (k)</td>
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<td>DAG</td>
<td>directed acyclic graphs</td>
<td>graph homomorphism</td>
</tr>
<tr>
<td>Mod</td>
<td>pairs ((R, M)) where (M) is module</td>
<td>module homomorphism</td>
</tr>
<tr>
<td></td>
<td>over the ring (R)</td>
<td></td>
</tr>
</tbody>
</table>
A map between categories

(Covariant) Functor

$F : C \to D$ consisting of

- For any $A \in Obj(C)$, $F(A) \in Obj(D)$
- For any $f \in Hom(A, B)$ where $A, B \in Obj(C)$, $F(f) \in Hom(F(A), F(B))$ such that
  - $F(f \circ g) = F(f) \circ F(g)$
  - $F(1_A) = 1_{F(A)}$
Functors

Arrows swap directions!

Contravariant Functor

$F : C \rightarrow \mathcal{D}$ consisting of

- For any $A \in \text{Obj}(C)$, $F(A) \in \text{Obj}(\mathcal{D})$
- For any $f \in \text{Hom}(A, B)$ where $A, B \in \text{Obj}(C)$, $F(f) \in \text{Hom}(F(B), F(A))$ such that
  - $F(f \circ g) = F(g) \circ F(f)$
  - $F(1_A) = 1_{F(A)}$

\[ \begin{array}{ccc}
A & \xrightarrow{g} & B \\
& \lrcorner \swarrow f \swarrow f \circ g & \xrightarrow{C} \\
\end{array} \quad \iff \quad \begin{array}{ccc}
F(A) & \xleftarrow{F(g)} & F(B) \\
& \lrcorner \swarrow F(f) \swarrow F(f \circ g) & \xleftarrow{F(C)} \\
\end{array} \]
Examples of Functors

- Forgetful functor $F : \text{Grp} \rightarrow \text{Set}$ that ignores the group structure and restriction on group homomorphism.

- Abelianization function $F : \text{Grp} \rightarrow \text{Ab}$ that maps $G \rightarrow G/[G, G]$.

- Homology operator $H_k : \text{Top} \rightarrow \text{Ab}$ that sends topological spaces to their $k$-dimensional homology groups and continuous maps to their maps induced by inclusion.

- Homotopy operator $\pi_k : \text{Top} \rightarrow \text{Grp}$ that sends topological spaces to their $k$-dimensional homotopy groups and continuous maps to their maps induced by inclusion.

- Cohomology operator $H^k : \text{Top} \rightarrow \text{Ab}$ that sends topological spaces to their $k$-dimensional homology groups and continuous maps to their maps induced by inclusion. This is a contravariant functor.
A (graph) filtration can be thought of as a functor $\text{DAG} \to \text{Top}$.

\[
\begin{array}{cccccc}
\cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow \\
\downarrow & & & & & \\
X_0 & \rightarrow & X_1 & \rightarrow & X_2 & \rightarrow X_3 \rightarrow X_4 \rightarrow X_5 \rightarrow X_6
\end{array}
\]

and the corresponding filtration of homology groups can also be thought of as a functor $\text{DAG} \to \text{Ab}$.

\[
\begin{array}{cccccc}
\cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow \\
\downarrow & & & & & \\
H_k(X_0) & \rightarrow & H_k(X_1) & \rightarrow & H_k(X_2) & \rightarrow H_k(X_3) \rightarrow H_k(X_4) \rightarrow H_k(X_5) \rightarrow H_k(X_6)
\end{array}
\]
Natural Transformations

Definition

If $\mathcal{C}$ and $\mathcal{D}$ are categories with functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\mu : F \to G$ satisfies

- For each $A \in \mathcal{C}$, there is a map $\mu_A : F(A) \to G(A)$
- For each morphism $f \in \text{Hom}_\mathcal{C}(A, B)$, $\mu_B \circ F(f) = G(f) \circ \mu_A$

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow \mu_A & & \downarrow \mu_B \\
G(A) & \xrightarrow{G(g)} & G(B)
\end{array}
\]
Given two categories \( C \) and \( D \) (Require objects of \( C \) to be sets) there is a category \([C, D]\) where

- Objects are covariant functors \( F : C \rightarrow D \)
- Morphisms are natural transformations \( \mu : F \rightarrow G \)

For example, \([\text{DAG}, \text{Top}]\) and \([\text{DAG}, \text{Ab}]\).
Persistent Homology as a Functor!

\[ PH_k : [\text{DAG, Top}] \to [\text{DAG, Ab}] \] is the functor that maps a filtration of spaces to the corresponding filtration of homology groups.

\[ X_0 \to X_1 \to X_2 \to X_3 \to X_4 \to X_5 \to X_6 \]

Apply the persistent homology functor and get:

\[ H_k(X_0) \to H_k(X_1) \to H_k(X_2) \to H_k(X_3) \to H_k(X_4) \to H_k(X_5) \to H_k(X_6) \]
If $F : \text{Top} \rightarrow \mathcal{C}$ is any functor, this induces a functor

$$PF : [\text{DAG, Top}] \rightarrow [\text{DAG, } \mathcal{C}]$$

- For example, $F = \pi_1$ induces persistent fundamental groups (or $F = \pi_k$ induces persistent homotopy).

$$P\pi_k : [\text{DAG, Top}] \rightarrow [\text{DAG, Grp}]$$

- $PH^k$ is persistent cohomology
- (Persistent) Alexander module

Note the filtration doesn’t need to be an interval graph, it can be any DAG.
What Properties of Persistent Homology Remain?

- Persistent homology groups?
- Barcodes? Persistence diagrams?
- Persistence modules?
- Stability?

For simplicity, we will assume the DAGs are all interval graphs

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Analogy of Persistent Homology Groups

\[ H^p_k(X_i) = \text{im}(H_k(X_i)) \to H_k(X_{i+p}) \]

If the functor \( F : \text{Top} \to C \) is a topological invariant, that is homeomorphic spaces have the same image, then

\[ F^p(X_i) = \text{im}(F(X_i)) \to \text{im}(F(X_{i+p})) \]

For non-interval graphs, can define this in terms of limits and co-limits.

Note: We need to assume \( C \) is a small category.
Persistence Diagrams

**Birth time**
There is a birth event at time \( \alpha \) if \( F(X_{\alpha-1} \to X_\alpha) \) is not an epimorphism.

**Death time**
If a birth event occurs at time \( \alpha \), the corresponding death event occurs at the smallest \( \beta \) such that \( \text{im}(F(X_{\alpha-1}) \to F(X_\beta)) = \text{im}(F(X_\alpha) \to F(X_\beta)) \).

Note: We need to assume \( \mathcal{C} \) is a small category and that the original filtration is topologically tame. (Only one topological change at any time.)

**Persistence diagram**
Consists of birth death pairs \((\alpha, \beta)\) in the plane.
Analogues of Persistence Modules

Modules have two main properties:

- Can add elements
- Can perform scalar multiplication over a base ring
- The operations have the correct unit, associate, commutative, and distributive properties.

Additive category

There is a functor $\bigoplus : C \times C \to C$.

Note: if $C$ is additive then so is $[\text{Top}, C]$.

Do we need scalar multiplication?
Unique decompositions

Work in a Krull-Schmidt category, an additive category such that every object is either

- Indecomposable or can be written as a finite direct sum of indecomposables
- And decompositions are unique: if $X_1 \oplus \cdots \oplus X_m \cong Y_1 \oplus \cdots \oplus Y_n$ then $m = n$ and there exists a permutation $\pi$ such that $X_{\pi(i)} \cong Y_i$.

When do these decompositions correspond to persistence diagrams?