Generalized Persistent Homology:
Part I, Modules

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Goals

- Understand the underlying structure of persistent homology
- Use more general collections of topological spaces, not just filtrations
- Do we have to use homology?
Filtration

\[ X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow \cdots \]

A sequence of topological spaces with maps (often inclusion) between them.
Note: can be indexed by rationals, reals, ...

Homology of Filtration

\[ H_k(X_0) \rightarrow H_k(X_2) \rightarrow \cdots \rightarrow H_k(X_n) \rightarrow \cdots \]

Persistent Homology

\[ H^p_k(X_t) = \text{im}(H_k(X_t) \rightarrow H_k(X_{t+p})) = \text{im}(f_{t,t+p}) \]

where \( f_{\alpha,\beta} : H_k(X_\alpha) \rightarrow H_k(X_\beta) \) is the map induced by the include \( X_\alpha \rightarrow X_\beta \).
Birth

An cycle \( c \in H_k(X_t) \) has birth time \( t \) if \( c \not\in \text{im}(H_k(X_s) \to H_k(X_t)) \) any \( s < t \).

Death

The death time of \( c \) is the smallest \( u \) such that the map \( f_{t,u} : H_k(X_t) \to H_k(X_u) \) maps \( u \) to 0.
Persistence Module

Definition

$\mathcal{P}H_k(\mathcal{X})$ is the submodule of $H_k(X_0) \oplus H_k(X_1) \oplus \cdots \oplus H_k(X_n)$ generated by elements of the form $(0, \ldots, 0, c, f_{\alpha,\alpha+1}(c), \ldots, f_{\alpha,\beta}(c) = 0, \ldots, 0)$ where $c \in H_k(X_\alpha)$ has birthtime $\alpha$.

Note: this is equivalent to the original definition (due to Carlsson and Zomordian) of the persistence module as a graded $\mathbb{F}[t]$-module.
Krull-Remak-Schmidt

**Theorem**

If $M$ is a Noetherian Artinian module the $M$ decomposes uniquely into direction summands

$$M \cong M_1 \oplus \cdots \oplus M_n$$

Recall that the standard persistence algorithm calculates birth and death pairs. Each of these pairs is a summand in the decomposition of the persistence module.

$$\mathcal{P}\mathcal{H}_k(X) = \bigoplus_i F(b_i, d_i)$$

where $F(b, d) = 0 \oplus \cdots \oplus F \oplus \cdots \oplus 0 \oplus \cdots \oplus 0$ has non-zero terms for $b \leq t < d$. 
\[ \mathcal{P}H_0(X) = F_{(0,\infty)} \]
\[ \mathcal{P}H_1(X) = F_{(1,4)} \oplus F_{(2,5)} \]
\[ \mathcal{P}H_2(X) = F_{(3,4)} \]
Quiver Representation

Quiver

A multi-digraph (Directed graph with multiple edges and loops)

Quiver Representation

Given a quiver $G = (V, E)$, a representation has

- A vector space $W_u$ for each $u \in V$
- A linear map $f : W_u \rightarrow W_v$ for each $(u, v) \in E$
Quiver Representation: Examples

**Standard Persistence**

\[ H_k(X_0) \rightarrow H_k(X_2) \rightarrow \cdots \rightarrow H_k(X_n) \rightarrow \cdots \]

The persistence module is a quiver representation.

**Zig-Zag Persistence (Carlsson-de Silva)**

\[ H_k(X_0) \leftrightarrow H_k(X_2) \leftrightarrow \cdots \leftrightarrow H_k(X_n) \leftrightarrow \cdots \]

Each arrow goes left or right.
Quiver Representation: Examples

Multi-dimensional Persistence (Carlsson-Zomordian)

\[
\begin{align*}
& H_k(X_{1,3}) \rightarrow H_k(X_{2,3}) \rightarrow H_k(X_{3,3}) \rightarrow \cdots \\
& H_k(X_{1,2}) \rightarrow H_k(X_{2,2}) \rightarrow H_k(X_{3,2}) \rightarrow \cdots \\
& H_k(X_{1,1}) \rightarrow H_k(X_{2,1}) \rightarrow H_k(X_{3,1}) \rightarrow \cdots
\end{align*}
\]
Quiver Representation: Examples

DAG Persistence (Chambers-L)

\[ H_k(X_1) \rightarrow H_k(X_5) \]
\[ H_k(X_3) \rightarrow H_k(X_6) \]
\[ H_k(X_2) \rightarrow H_k(X_4) \rightarrow H_k(X_k) \]
Gabriel’s Theorem

**Theorem**

If the underlying undirected graph is an ADE Dynkin diagram that there are finitely many possible irreducible submodules of a quiver representation.

Standard and zig-zag is a Type A Dynkin diagram and irreducible submodules are all of the form $\mathbb{F}^{(b,d)}$. 
Decompositions in DAG Persistence