CS 314 - Network Flow

Announcements

- Turn in HW
- Next HW is posted (written, so due next Wed.)
- Office hours tomorrow changed
  - 10(ish) to noon
Goal: Model transportation networks
(from "Secret" government pub in 1955)
More formally:

- A directed graph $G = (V, E)$
- Each edge has a maximum capacity $c_e$
- Two special vertices $s, t \in V$
  - $s$ is the source
  - $t$ is the sink

Note: $s$ has no incoming edges, and $t$ has no outgoing edges.
Think of edges as pipes, roads, network connections, etc...

The goal is to "push" as much flow from $s$ to $t$.

$$V(F) = 20 + 10 = 30$$
Formally:

A flow is a function \( f : E \rightarrow \mathbb{R}^+ \) (some amount sent along each edge) such that:

1. Capacity constraint: \( \forall e \in E, \ 0 \leq f(e) \leq c_e \)

2. Conservation constraint: \( \forall v \in V, \text{ if } v \neq s \text{ or } t \)
   \[
   \sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e) \quad \text{for } v \neq s \text{ or } t
   \]

\[ f_{\text{in}}(v) \quad f_{\text{out}}(v) \]
Notation: for any \( S \subseteq U \),
\[
\text{f}^{\text{out}}(S) = \sum_{e \text{ out of } S} f(e)
\]
(\( \text{f}^{\text{in}}(S) \) similarly)
Maximum Flow Problem

- The value of a flow is $\sum_{e \text{ out of } s} f(e) = \sum_{e \text{ into } t} f(e)$

**Goal:** Find flow with maximum value.

(Arrange the traffic as efficiently as possible.)
Basic obstacle

For any $S \subseteq V$ with $s \in S$, $t \in V - S = T$, all flow must leave $S$ and enter $T$.

So flow $\leq$ sum of edge capacities from $S$ to $T$ (this is called $(S,T)$-cut)
Computing Flow

Ideas?

And an s to t path + push as much flow as we can

Now- no s to t paths
Problem: We can get stuck!

So we may need to "unpush" flow.

Def: The residual graph $G_f$ of $G$ with respect to a flow $f$ is a graph with:

- $G_f$ has the same vertex set as $G$
- For each edge $(u,v)$ in $G$ with $f(u,v) < C_{uv}$, add an edge to $G_f$ from $u$ to $v$ with weight $C_{uv} - f(u,v)$
- If $f(e) > 0$ where $e = (u,v)$, add edge $vu$ in $G_f$ of value $f(e)$
Ex:

\[ G + f \]

\[ G_f \]
So $G_f$ does have an $s$-to-$t$ path!

(Notice that path “unpushes” some flow.)

We find $s$-to-$t$ path in $G_f$ by either increase or decrease flow along each edge in that path.
Claim: New flow $f'$ is a valid flow.

pf: Need to verify 2 things:

1. Capacity constraint: only changed flow for edges on path $P$.
   
   Let $e = (uv) \in P$.
   
   $w(\text{bottleneck edge on } P) = (c_e - f(e))$

   Let bottleneck edge in $P$ = min weight edge of $P$ in $G_f$.

   So adding $w(\text{bottleneck})$ to every edge in $P$ cannot exceed capacity of each edge.
2. Conservation before, $in = out$ for every vertex.

The only flow change is along $P$.

Change each vertex in $P$ along 2 of its edges, by the same amount.

So flow in is still = flow out.
Our Algorithm: [Ford-Fulkerson 1956]
- Find a path from s to t in G_f
- Push flow along s-t path
- Repeat until G_f contains no s-t paths
Pseudo code

MaxFlow(G):
\[ f(e) \leftarrow 0 \quad \text{for } e \in E \]
\[ G_f \leftarrow G \]

While there is an s-t path in \( G_f \):
\[ P \leftarrow \text{s-t path in } G_f \]
\[ f' = \text{Augment}(f, P) \]
\[ f \leftarrow f' \]
Update \( G_f \)

Return \( f \)
Augment \((f, P)\):
\[\begin{align*}
&b \leftarrow \text{bottleneck edge of } P \\
&\text{for each edge } (u, v) \in P \\
&\text{if } e = (u, v) \text{ is forward in } G \\
&\quad f(e) \leftarrow f(e) + b \\
&\text{if } e = (v, u) \text{ is backwards in } G \\
&\quad f(e) \leftarrow f(e) - b \\
&\text{return } f
\end{align*}\]
We know this returns a valid flow, but haven't shown it returns the maximum flow.

**Def:** An \( s-t \) cut \( (S, T) \) is a partition of \( V \) into 2 sets \( S, T \) with \( s \in S \) and \( t \in T \).

The capacity of a cut \( c(S, T) = \sum_{e \text{ out of } S} c_e \)
Strategy: 2 things

1. Thm: let \( f \) be any s-t flow, and \((S, T)\) any s-t cut.
   Then \( v(f) = c(S, T) \)

2. Given a flow \( f \) where there is no s-to-t path in \( G_f \), we can find a cut \((S^*, T^*)\) with:
   \[ v(f) = c(S^*, T^*) \]
   max flow, min cut
First, a lemma:

**Lemma:** Let $f$ be any $s-t$ flow, and $(S, T)$ any $S-T$ cut. Then $v(F) = f^{out}(S) - f^{in}(S)$.

**Proof:**