Announcements

- HW due

- Next HW will be posted today due next Friday
7.6: Disjoint Paths in Directed and Undirected graphs

Def: Two paths are edge-disjoint if they do not have any edges in common.

Goal: Find the maximum number of edge disjoint paths between 2 vertices $s$ and $t$. 
Given a directed graph $G$, how can we reduce this to a flow problem?

* - Give every edge capacity 1.

Find max flow in $G$.

Return that value.
Claim: $G$ has $k$ edge-disjoint $s$-$t$ paths

$\iff G'$ has a flow of value $k$.

pf: $\Rightarrow$: Given $k$ $s$-$t$ paths, send 1 unit of flow along each.

$\Leftarrow$: Assume we have a flow in $G'$ of value $k$. 
pf cont: Induction on # edges that carry flow.

Base case: If \( v = 0 \), done.

Inductive step: If \( v \geq 1 \), we can find an edge out of \( s \) that carries 1 unit of flow.

Trace a set of edges which all carry one unit of flow (since flow is conserved).

Either we repeat a vertex or we hit \( t \).

If we hit \( t \), have a path.

Make this one of \( s \rightarrow t \rightarrow \) paths and reset flow on those edges to 0.
If we repeat a vertex, consider the set of edges between those 2 vertices which have flow = 1.

Create a new flow $f'$ which is the same as $f$ except all edges on cycle are now $f'(e) = 0$.

Claim: $f'$ is a valid flow with the same value + fewer edges (so It covers it).

1. Capacity constraint: now carry 0 flow (still $\leq 1$)
2. Conservation constraint: for any vertex on this cycle, reduce $f^+$ and $f^-$ by 1.

$\nu(f') = \nu(f)$ since we haven't changed any edge out of $S$. \[\]
Runtime:

\[ O(mC) = O(mn) \]
\[ O(mn^2) \]
\[ O(n^3) \]
\[ O(m^2 \log C) \]

Conversion: \[ O(m) \]

Total: \[ O(mn) \]
What about undirected graphs?

Convert to directed graphs by drawing 2 edges (1 in each direction) for every edge.
A few extensions (7.7)

What if we have multiple sources & sinks?

Let $d_v =$ demand at node $v$.

- if $d_v < 0$, it is a source
- if $d_v > 0$, it is a sink

Goal: For every $v$, $d_v = f^{in}(v) - f^{out}(v)$

(Call this a circulation.)
How can we reduce to a normal flow problem? add $s^* + t^*$

$G'$

Need flow in $G'$ of value $\sum_{v \in \text{sources}} d_v$
Circulation in $G$ of value $k$ \[\Rightarrow\] flow in $G'$ of value $k$
Adding lower bounds

Suppose we want flow on each edge to meet a certain lower bound, also.

So want flow with:

\[ \forall e \in E, \quad l_e \leq f(e) \leq c_e \]

\[ \implies (2) \quad f_{in}(v) - f_{out}(v) = d_v \]

\[ \text{same as with circulations} \]
To start, put le units of flow on each edge, so \( f_0(e) = le \). Satisfies (1) but not (2).

Have: \( f_0^\text{in}(v) - f_0^\text{out}(v) = \sum_{e \text{ into } v} le - \sum_{e \text{ out of } v} le \)

Call this \( L_v \).

If \( L_v = \delta_v \), done.
If not - need to get \((d_v - L_v)\) more into \(v\)!
(Note - no lower bound!)

So create \(G'\) with same \(V\) & \(E\):
* Each edge now has capacity "be".
* Each node has demand \(d_v - L_v\).

This is a circulation!
Need circulation in 6 (with lower bounds met) \Rightarrow \text{circulation in } 6'

(see book)