Announcements

- HW due next Wednesday
Definition:

Let \( \text{OPT}(X) \) be the value of an optimal solution, and \( A(X) \) be the value of an algorithm \( A \)'s solution (on input \( X \)).

\( A \) is an \( \alpha(n) \)-approximation if and only if

\[
\frac{\text{OPT}(X)}{A(X)} \leq \alpha(n) \quad \text{and} \quad \frac{A(X)}{\text{OPT}(X)} \leq \alpha(n)
\]

for all inputs \( X \) of size \( n \).

Last time: \( \frac{3}{2} \)-approx
Usually, only one of these inequalities is important,

\[
\frac{\text{OPT}(X)}{A(X)} \leq \alpha(n) \quad \text{and} \quad \frac{A(X)}{\text{OPT}(X)} \leq \alpha(n)
\]

\[\Rightarrow\] minimization problems

\[\Rightarrow\] maximization problems
Last time:

We found a schedule where $T \leq \frac{3}{2} T^*$.

So $\frac{T}{T^*} \leq \frac{3}{2}$ and $\frac{T^*}{T} \leq \frac{2}{3}$.

This means we had a $\frac{3}{2}$-approximation.

(we wanted to minimize $T$'s value)
Vertex Cover
- A set of vertices which "covers" every edge in the graph.

Note: this will be a minimization problem - want V.C. as small as possible.
What's a natural approximation method?

Greedy

Pick someone with a large # of uncovered edges.
Pseudocode:

**GREEDYVERTEXCOVER(G):**

\[ C \leftarrow \emptyset \]

while \( G \) has at least one edge

\[ v \leftarrow \text{vertex in } G \text{ with maximum degree} \]

\[ G \leftarrow G \setminus v \]

\[ C \leftarrow C \cup v \]

return \( C \)
Thm: Greedy Vertex Cover gives an $O(\log n)$-approximation.

Proof:

Notation: Let $G_i$ be the graph after $i$ loop iterations, and $d_i^*$ be maximum degree in $G_{i-1}$.

Also: $C^*$ is optimal cover,

\[ \# \text{edges in } G_i \]
pf (cont.):

\[
\text{GREEDYVERTEXCOVER}(G): \\
C \leftarrow \emptyset \\
G_0 \leftarrow G \\
i \leftarrow 0 \\
\text{while } G_i \text{ has at least one edge} \\
i \leftarrow i + 1 \\
v_i \leftarrow \text{vertex in } G_{i-1} \text{ with maximum degree} \\
d_i \leftarrow \text{deg}_{G_{i-1}}(v_i) \\
G_i \leftarrow G_{i-1} \setminus v_i \\
G_i \leftarrow G_{i-1} \setminus v_i \\
C \leftarrow C \cup v_i \\
\text{return } C
\]
more proof: $C^*$ is also a vertex cover for $G_{i-1}$, so

$$\sum_{v \in C^*} \deg_G(v) \geq |E(G_i)|$$

$\Rightarrow$ average degree of a vertex in $C^*$

$$\geq \frac{|E(G_{i-1})|}{|C^*|}$$

$\Rightarrow$ $d_i^* \geq \frac{|E(G_{i-1})|}{|C^*|}$ (since $d_i^*$ is maximum degree)

For any $j \geq i-1$, $G_j$ has fewer edges than $G_{i-1}$

so $d_i^* \geq \frac{|E(G_j)|}{|C^*|}$
\[ \text{OPT} = |C^*| \]

\[
\sum_{i=1}^{\text{OPT}} d^i \geq \sum_{i=1}^{\text{OPT}} \frac{|G_{i-1}|}{\text{OPT}} \geq \sum_{i=1}^{\text{OPT}} \frac{|E(G_{\text{OPT}})|}{\text{OPT}}
\]

\[ \text{All in } \mathbb{D} \quad \text{All in } \mathbb{D} \]

And

\[
\sum_{i=1}^{\text{OPT}} \frac{|E(G_{\text{OPT}})|}{\text{OPT}} \geq |E(G_{\text{OPT}})| = |E(G)| - \sum_{i=1}^{\text{OPT}} d^i
\]

Rewrite:

\[
\sum_{i=1}^{\text{OPT}} d^i \geq |E(G)| - \sum_{i=1}^{\text{OPT}} d^i
\]

\[ 2 \left( \sum_{i=1}^{\text{OPT}} d^i \right) \geq |E(G)| \Rightarrow \sum_{i=1}^{\text{OPT}} d^i \geq \frac{|E(G)|}{2} \]
\[
\sum_{i=1}^{\text{OPT}} d_i \geq \frac{\log(m)}{2}
\]

So I've deleted half the edges in OPT repetitions of the loop.

So after \( \text{OPT}(\log m) \leq 2 \text{OPT} \) iterations, all edges are gone.
Unfortunately, this can't be improved.

There is a nice (recursive) construction of a graph of size $n$ for which greedy returns a vertex cover of size $\frac{1}{2}OPT \cdot \log n$. 
Don't always be greedy!

Another idea -
  take any edge \( e=(u,v) \) in \( G \).
  What must be in any vertex cover?
    either \( u \) or \( v \) is in \( C^* \)
    so add both
Pseudo code

DUMBVERTEXCOVER(G):
    C ← ∅
    while G has at least one edge
        (u, v) ← any edge in G
        G ← G \ {u, v}
        C ← C ∪ {u, v}
    return C
Dumb Vertex Cover is a 2-approximation!

$C^*$ contains one of the 2 vertices I add in every loop iteration.

$\Rightarrow |C^*| \leq \frac{1}{2}|C|$
Next time: Traveling Salesman

Q: Is there a Hamiltonian cycle in a weighted (complete) graph with length ≤ k?

NP-Hard, Why?

Hand you G a ask if there is a Hamilton cycle.
Take $G$ and make all edges of weight $= 1$.
Any other edge, give larger weight say $\leq n$.

Ask: Is there a TS tour of length $\leq n$?