Announcements

- HW out, due (oral grading) next Tuesday

- Next HW will be out Mon./Tues. next week, & will be written NP-Completeness problems.
**Vertex Cover**

**Def.** A vertex cover is a subset of vertices $S \subseteq V$ such that every edge is adjacent to a vertex in $S$.

**Ex.**

![Diagram of a vertex cover](image)
Q: Given a graph $G$ and value $k$, is there a vertex cover of size $\leq k$?

Vertex Cover is NP-Complete:

1. VC is in NP:
   - Given $k$ vertices, label edges $1 \ldots m$.
   - Mark edges in $G$ that are adjacent to one of the $k$ vertices.
   - If all edges are marked, the $k$ vertices are a vertex cover.
   - $O(m \times k)$
2. \text{V.C. is NP-Hard: } \begin{array}{l}
\text{I.S. } \leq_{P} \text{V.C.}
\end{array}

\text{Key Fact: If } I \text{ is an ind. set, then } V - I \text{ is a vertex cover.}

\text{pf: } \text{Supp I is an ind. set in a graph G. So no edges b/t any 2 vertices in I.}
\Rightarrow \text{every edge has an endpoint not in I.}
\Rightarrow V - I \text{ is a vertex cover. \hfill \square}
Ind. Set ≤p Vert. Cover:

Input: \( G, k \) want to say yes if \( G \) has ind. set of size \( \geq k \).

\[ \leq \]

Transform to another \( G' \) as ask is \( IS \) of size \( \leq k' \):

\( G \) to have IS of size \( \geq k \)

\[ \iff \]

\( G' \) has V.C. of size \( \leq k' \)
So we don't need to transform $G$ at all!

Given $G$ and $k$ as input to ind. set problem, just ask black box for vertex cover if there is a vertex cover of size $n-k$.

If ind set of size $\leq k$, then v.c. of size $\leq n-k$.

If v.c. of size $\leq n-k$, then ind set of size $\geq k$.

So here, set $G' = G$

$k' = n-k$
Recap:

Vertex Cover in NP-Complete.

Set $k' = n - k$

graph $G = (V, E)$ \[ \xrightarrow{\text{trivial}} \] graph $G = (V, E)$ \[ \xrightarrow{\text{MinVertexCover}} \] smallest vertex cover $S$

largest independent set $V \setminus S$
Hamiltonian Cycle

A Hamiltonian cycle is a cycle in a graph which visits every vertex exactly once.

Ex:

Has no Hamiltonian cycle.
Hamiltonian cycle is NP-Complete

1. Ham. cycle is in NP
   Given a cycle, it visits every vertex once
   \[O(n)\]

2. Reduce \underline{3SAT} to Ham. cycle.
   (In lec notes, he reduces Vertex Cover to Ham. cycle.)

From Ch. 8
Setup:
We'll show \( \text{3SAT} \leq \text{Ham. cycle} \).

So we have a "black box" which, given a graph \( G \), will output yes if \( G \) contains a Hamiltonian cycle.

Our input is a 3SAT problem.
\[
(x_1 \lor \overline{x_2} \lor x_3) \land (x_2 \lor x_5 \lor \overline{x_6}) \land \ldots
\]

We need to (in polynomial time) translate that formula into a graph.
$G$ will have $2^n$ different Hamilton cycles possible (1 for each truth assignment)

\[
\text{Variables } x_1 \ldots x_n \\
\text{Clauses } c_1 \ldots c_k \\
\big( c_1 \big) \wedge \big( c_2 \big) \wedge \big( c_3 \big) \wedge \ldots \big( c_k \big)
\]

Each variable will become a path.
- variable gadgets

Each clause will be a vertex.
Variable gadgets

If I go left to right on $P_i$, then $x_i$ will be "true".

$x_1 \rightarrow P_1$:

$x_2 \rightarrow P_2$:

$\vdots$

$P_n$:

$3k+2$ vertices on each path $P_i$
Clause gadget vertex

Each clause gets 1 vertex + we hook it to the 3 relevant paths: π

Clause 1: \( x_i \land \overline{x_j} \land x_k \)

\( P_i \rightarrow \pi \rightarrow C_1 \), (if \( x_i \) is false, then \( \overline{x_j} \) is true)

\( P_j \rightarrow \pi \rightarrow C_2 \), (\( C_2 \) has \( x_k \) in it)

\( P_k \rightarrow \pi \rightarrow C_1 \)
Claim: 3SAT instance is satisfiable
\[ \iff \exists \text{ ham. cycle in } G \]

**Proof:**
- Suppose there is an assignment of \( x_1, \ldots, x_n \) that evaluates to true.
- In \( G \), start at a \& traverse each \( P_i \) left to right if \( x_i \) is true, right to left if \( x_i \) is false.
- Since formula evaluates to true, at least one variable in each clause is true.
- In some cycle, I'll visit vertex \( c_i \) in the upath which corresponds to the true variable.
Let Spps Ham cycle C in G.

Must use b \rightarrow a edge.

Then must go to P_i.

Then P_2

Then P_3

(All details in Ch. 8)
How long did transformation take?

$O(nk)$
Travelling salesman Problem

Suppose we have n cities connected by a road network. (complete, directed graph)

Want a tour which visits every city and is as short as possible.

Q: Given a graph G and value k, is there a tour with cost \( \leq k \)?
TSP is NP-Complete