Recurrence Relations: in-class exercises

For each of the following:

- Determine a tight upper bound for the recurrence, either using the Master theorem or a recursion tree.
- Use the substitution method to formally prove the upper bound using induction. (Remember you may need to strengthen the inductive hypothesis from the obvious choice.)
- If you have extra time, you can consider proving asymptotic lower bounds for those same recurrences (e.g., prove the $\Omega(\cdot)$ bound by induction).

For review, the Master theorem considers recurrence relations of the following form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

for $a \geq 1$ and $b \geq 1$. The three cases state:

1) If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$.

2) If $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a \lg n}\right)$.

3) If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some constant $\epsilon > 0$ and if $a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large $n$, then $T(n) = \Theta(f(n))$.

Instructor’s Examples

i) $T(n) = 2T\left(\frac{n}{2}\right) + n$. We briefly consider why the invalid inductive hypothesis that $T(n) \leq c \cdot n$ is fatally flawed.
ii) \( T(n) = 2T\left(\frac{n}{2}\right) + n^2 \). Solves to \( T(n) = \Theta(n^2) \), using typical inductive hypothesis.

Inductive hypothesis: \( T(n) \leq c \cdot n^2 \).
Base case: trivial for large enough \( c \).
Inductive Case:

\[
T(n) = 2T\left(\frac{n}{2}\right) + n^2 \\
\leq 2 \left( c \cdot \left(\frac{n}{2}\right)^2 \right) + n^2 = \frac{1}{2}c \cdot n^2 + n^2 \\
\leq c \cdot n^2 \quad \text{for } c \geq 2
\]

iii) \( T(n) = 4T\left(\frac{n}{2}\right) + n \). Solves to \( T(n) = \Theta(n^2) \), however the inductive hypothesis \( T(n) \leq c \cdot n^2 \) fails. We strengthen hypothesis to \( T(n) \leq c \cdot n^2 - d \cdot n \) for some \( d \).

Inductive hypothesis: \( T(n) \leq c \cdot n^2 - d \cdot n \).
Base case: trivial for large enough \( c \).
Inductive Case:

\[
T(n) = 4T\left(\frac{n}{2}\right) + n \\
\leq 4 \left( c \cdot \left(\frac{n}{2}\right)^2 - d \cdot \frac{n}{2} \right) + n \\
= c \cdot n^2 - 2d \cdot n + n = c \cdot n^2 - dn - n(d - 1) \\
\leq c \cdot n^2 - d \cdot n \quad \text{for } d \geq 1
\]

Student Exercises
A) \( T(n) = T\left(\frac{n}{2}\right) + 1 \).
B) \( T(n) = 2T\left(\frac{n}{2}\right) + 1 \).
C) \( T(n) = 4T\left(\frac{n}{3}\right) + n \).
D) \( T(n) = 4T\left(\frac{n}{2}\right) + n^2 \)
E) \( T(n) = 7T\left(\frac{n}{2}\right) + n^2 \)
F) \( T(n) = 3T\left(\frac{n}{4}\right) + n \log n \)
G) \( T(n) = T\left(\frac{7}{10}n\right) + T\left(\frac{1}{5}n\right) + n \)
H) \( T(n) = 2T\left(\sqrt{n}\right) + \log n \). (Hint: consider substitution \( m = \log n \).)
I) \( T(n) = 3T\left(\sqrt{n}\right) + \log n \).