CS314: Algorithms
Homework 5

1. For each of the following two statements, decide if it is true or false. If true, give a short explanation/proof, and if false, give a counterexample.

   (a) Suppose we are given an instance of the minimum spanning tree problem on a graph G, with all positive and distinct edge costs. Let T be a minimum spanning tree for this instance. Now suppose we replace each edge weight $w(e)$ by its square, $w(e)^2$, which has the same edge and vertex sets but a different weight function. True or false: T is still a minimum spanning tree for this new graph.

   (b) Now suppose we have a weighted directed graph G with all positive distinct edge weights. Let P be a minimum cost path from vertex s to t in this graph. We now again replace the weight on each edge $w(e)$ with its square $w(e)^2$, creating a new graph with the same edge and vertex sets but new weights. True or false: P must still be a minimum cost s to t path in this new graph.

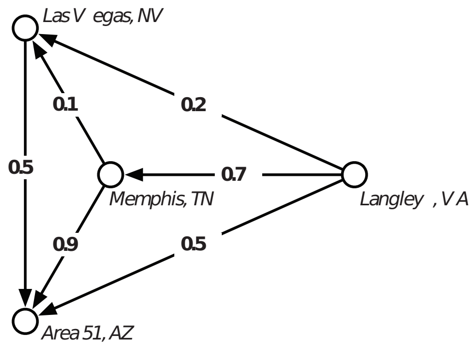
2. Let $G = (V, E)$ be an undirected graph with a weight $w(e)$ for each edge $e \in E$. Assume you are given a minimum spanning tree T for G. Now assume that a single new edge e is added to G, connecting two nodes u and v and with a cost c.

   (a) Give an efficient algorithm to test if T is still a minimum spanning tree for this new graph. Make your algorithm run in $O(m+n)$ time. Can you do it in $O(n)$ time? Please be sure to note any assumptions you make about what data structure is used to represent both T and G.

   (b) Suppose T is no longer the minimum spanning tree for G. Give an algorithm that is as fast as possible to update T to the new minimum spanning tree.

3. Mulder and Scully have computed, for every road in the United States, the exact probability that someone driving on that road wont be abducted by aliens. Agent Mulder needs to drive from Langley, Virginia to Area 51, Nevada. What route should he take so that he has the least chance of being abducted?

   More formally, you are given a directed graph $G = (V, E)$ where every edge e has an independent safety probability $p(e)$. The safety of a path is the product of the safety probabilities of its edges. Design and analyze an algorithm to determine the safest path from a given start vertex s to a given target vertex t.



For example, with the probabilities shown above, if Mulder tries to drive directly from Langley to Area 51, he has a 50% chance of getting there without being abducted. If he stops in Memphis, he has a $0.7 \times 0.9 = 63\%$ chance of arriving safely. If he stops first in Memphis and then in Las Vegas, he has a $1 - 0.7 \times 0.1 \times 0.5 - 96.5\%$ chance of being abducted! (That's how they got Elvis, you know.)