Some Midterm Exercises

1. In a smooth m-manifold M, show every point $p \in M$ has a chart (U, φ) with $\varphi(p) = \mathbf{0}$ and $\varphi(U) = B(\mathbf{0}, 1)$, the open ball of radius 1 around $\mathbf{0} \in \mathbb{R}^m$.

2. Let $f : M \to N$ be a smooth map of manifolds. Define the graph of f to be $\Gamma(f) \subset M \times N$ as $\Gamma(f) = \{(x, y) \in M \times N | f(x) = y\}$. Show $\Gamma(f)$ is a manifold.

3. Define $\sigma : M \times M \to M \times M$ by $\sigma(x, y) = (y, x)$. Show that σ is a diffeomorphism.

4. Given points $x_1, \ldots, x_k \in M$, and values $v_1, \ldots, v_k \in \mathbb{R}$, show there is a smooth function $f : M \to \mathbb{R}$ with $f(x_i) = v_i$ for all i.

5. In homogeneous coordinates on $\mathbb{R}P^1$, every point but $[1 : 0]$ can be written as $[x : 1]$, and every point but $[0 : 1]$ can be written as $[1 : y]$. Away from those two points, write $\frac{\partial}{\partial x}$ in terms of $\frac{\partial}{\partial y}$.

6. On the torus $\mathbb{T}^2 = S^1 \times S^1 = \{(e^{i\theta}, e^{i\phi})\}$, define a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$f(e^{i\theta}, e^{i\phi}) = (e^{i(a\theta + b\phi)}, e^{i(c\theta + d\phi)})$$

   where a, b, c, d are integers. Show that f is well defined, and is a diffeomorphism if $ad - bc = \pm 1$.