Math 642 # Midterm Review Problems

1. Show that every manifold has a nonzero complete vector field.
2. Let $f : M \to N$ be a smooth map of smooth manifolds, let $\sigma : [a,b] \to M$ be a curve in M, and let ω be a one-form on N. Show that
$$\int_\sigma f^*\omega = \int_{f\circ\sigma} \omega.$$
3. Suppose σ is a locally conservative 1-form on S^2. Show there is $f \in C^\infty(S^2)$ with $\sigma = df$.
4. Let M be a manifold, and $x, y \in M$. Show that for any $D > 0$, there is a Riemannian metric g on M with $d_g(x,y) = D$.
5. Define a helix $H \subset \mathbb{R}^3$ parametrically by $(r\cos(\theta), r\sin(\theta), \theta)$ for $r \in [0,\infty]$ and $\theta \in \mathbb{R}$. Calculate the induced metric on H.
6. Given a Riemannian metric g on the circle S^1, define the $L(g)$ to be the length (using g) of the curve that goes once around the circle. Show that any two metrics g, h on S^1 with $L(g) = L(h)$ are isometric.
Hint: map to the canonical circle C of length L, where $C = [0,L]/(L \sim 0)$ with metric dt^2.
7. Given a one form $\omega \in \mathcal{T}^1(M)$ and a vector field X, define $\mathcal{L}_X\omega$ by
$$(\mathcal{L}_X\omega)(Y) = \omega([X,Y]) - X.\omega(Y).$$
Show that $\mathcal{L}_X\omega$ is a tensor.
8. Define an r-covariant tensor σ on $\mathbb{R}^2$ by summing 2^r terms:
$$\begin{aligned}\sigma = {} & dx \otimes dx \otimes \cdots \otimes dx \otimes dx \\ & + dx \otimes dx \otimes \cdots \otimes dx \otimes dy \\ & + dx \otimes dx \otimes \cdots \otimes dy \otimes dx \\ & \cdots \\ & + dy \otimes dy \otimes \cdots \otimes dy \otimes dy\end{aligned}$$
where the sum is over all possible choices of dx and dy in each r-fold product term.
Find $\iota^*(\sigma)$, where ι is the inclusion map $S^1 \to \mathbb{R}^2$.