# Midterm Review Problems

1. Show that every manifold has a nonzero complete vector field.

**Solution:** Let $x \in M$ be any point, and choose a coordinate neighborhood $U$ of $x$ and a cutoff function $f$ on $M$ which is 1 at $x$ and has compact support in $U$. Let $Y$ be any nonzero vector field defined on $U$ (for example, $Y = \frac{\partial}{\partial x^1}$, where $x^1$ is the first coordinate function). Define $X = fY$ on $U$, and extend to all of $M$ by zero. Then $X$ is a nonzero, smooth vector field on $M$. Because $X$ is compactly supported, it is complete.

2. Let $f : M \to N$ be a smooth map of smooth manifolds, let $\sigma : [a, b] \to M$ be a curve in $M$, and let $\omega$ be a one-form on $N$. Show that

$$\int_\sigma f^*\omega = \int_{f\circ\sigma} \omega.$$

**Solution:**

$$\begin{aligned} \int_\sigma f^*\omega &= \int_a^b (f^*\omega)(\sigma'(t))dt \\ &= \int_a^b \omega(Tf\sigma'(t))dt \\ &= \int_a^b \omega((f\circ\sigma)'(t))dt \\ &= \int_{f\circ\sigma} \omega. \end{aligned}$$

3. Suppose $\sigma$ is a locally conservative 1-form on $S^2$. Show there is $f \in C^\infty(S^2)$ with $\sigma = df$.

**Solution:** Since $S^2$ is simply connected, any closed curve $\gamma$ on $S^2$ is homotopic to a point. Because $\sigma$ is locally conservative, path integrals are homotopy invariant so that $\int_\gamma \sigma = 0$. Then $\sigma$ is conservative, hence exact, hence equal to $df$ for some $f$. More specifically, one could fix $x_0 \in M$ and define $f(x) = \int_\gamma \sigma$ where $\gamma$ is any curve joining $x_0$ to $x$.

4. Let $M$ be a manifold, and $x, y \in M$. Show that for any $D > 0$, there is a Riemannian metric $g$ on $M$ with $d_g(x, y) = D$.

**Solution:** Let $h$ be any Riemannian metric on $M$, and put $\lambda = d_h(x,y) > 0$. Define $g = \frac{D^2}{\lambda^2}h$ (which is easily seen to be a Riemannian metric). Then for any curve $\gamma : [a,b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$, we have

$$\text{len}_g(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}dt = \int_a^b \sqrt{\frac{D^2}{\lambda^2}h(\gamma'(t), \gamma'(t))}dt = \frac{D}{\lambda}\,\text{len}_h(\gamma)$$

Since $\lambda = d_h(x,y)$ is the infimimum over all curves $\gamma$ of $\text{len}_h(\gamma)$, the infimum of $\text{len}_g(\gamma)$ is $d_g(x,y) = \frac{D}{\lambda}d_h(x,y) = D$.

5. Define a helix $H \subset \mathbb{R}^3$ parametrically by $(r\cos(\theta), r\sin(\theta), \theta)$ for $r \in [0,\infty]$ and $\theta \in \mathbb{R}$. Calculate the induced metric on $H$.

**Solution:** $dx = \cos(\theta)dr - r\sin(\theta)d\theta$, $dy = \sin(\theta)dr + r\cos(\theta)d\theta$, and $dz = d\theta$, so

$$dx^2 + dy^2 + dz^2 = dr^2 + (1+r^2)d\theta^2$$

which gives the metric on $H$.

6. Given a Riemannian metric $g$ on the circle $S^1$, define the $L(g)$ to be the length (using $g$) of the curve that goes once around the circle. Show that any two metrics $g, h$ on $S^1$ with $L(g) = L(h)$ are isometric.
Hint: map to the canonical circle $C$ of length $L$, where $C = [0,L]/(L \sim 0)$ with metric $dt^2$.

**Solution:** Conisder $S^1$ with parameter $\theta \in [0, 2\pi)$. Write $g = c(\theta)d\theta^2$ for some smooth function $c(\theta)$ which is periodic with period $2\pi$. Since $g$ is positive definite, $c$ is positive, so put $f(\theta) = \sqrt{c(\theta)}$, a positive, smooth periodic function with $g = f(\theta)^2 d\theta^2$. Define

$$F(\theta) = \int_0^\theta f(t)dt.$$

Then

$$F(\theta + 2\pi) - F(\theta) = \int_\theta^{\theta+2\pi} f(\theta)d\theta = \int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} \sqrt{g(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta})}d\theta = L(g) = L$$

which shows that $\theta \to t = F(\theta)$ gives a well defined map from $S^1$ to $C$. Now

$$dt = dF(\theta) = F'(\theta)d\theta = f(\theta)d\theta$$

which shows that $F$ takes $g$ to $dt^2$, so that $(S^1, g)$ and $(C, dt^2)$ are isometric. Since $(S^1, g)$ and $(S^1, h)$ are both isometric to $(C, dt^2)$, they are isometric to each other.

7. Given a one form $\omega \in \mathcal{T}^1(M)$ and a vector field $X$, define $\mathcal{L}_X\omega$ by

$$(\mathcal{L}_X\omega)(Y) = \omega([X,Y]) - X.\omega(Y).$$

Show that $\mathcal{L}_X\omega$ is a tensor.

**Solution:** For any vector field $Y$ and $f \in C^\infty(M)$, we have:

$$\begin{aligned}
\mathcal{L}_X\omega(fY) &= \omega([X, fY]) - X\omega(fY) && (1)\\
&= \omega(X(fY) - (fY)X) - X(f\omega(Y)) && (2)\\
&= \omega((Xf)Y + fXY - fYX) - (Xf)\omega(Y) - fX\omega(Y) && (3)\\
&= (Xf)\omega(Y) + f\omega([X, Y]) - (Xf)\omega(Y) - fX\omega(Y) && (4)\\
&= f\mathcal{L}_X\omega(Y) && (5)
\end{aligned}$$

Since $\mathcal{L}_X\omega$ is $C^\infty(M)-$linear, it's a tensor.

8. Define an $r$-covariant tensor $\sigma$ on $\mathbb{R}^2$ by summing $2^r$ terms:

$$\begin{aligned}
\sigma = {} & dx \otimes dx \otimes \cdots \otimes dx \otimes dx\\
& + dx \otimes dx \otimes \cdots \otimes dx \otimes dy\\
& + dx \otimes dx \otimes \cdots \otimes dy \otimes dx\\
& \cdots\\
& + dy \otimes dy \otimes \cdots \otimes dy \otimes dy
\end{aligned}$$

where the sum is over all possible choices of $dx$ and $dy$ in each $r$-fold product term.
Find $\iota^*(\sigma)$, where $\iota$ is the inclusion map $S^1 \to \mathbb{R}^2$.

**Solution:** Replace $dx = \cos\theta d\theta$ and $dy = \sin\theta d\theta$, then use the binomial theorem to get $(\cos(\theta) + \sin(\theta))^r d\theta \otimes \cdots \otimes d\theta$. A similar approach is to notice that $\sigma = (dx + dy)^r$.