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Recurrence Relations: in-class exercises

For each of the following:

- Determine a tight upper bound for the recurrence, either using the Master theorem or a recursion tree.
- Use the substitution method to formally prove the upper bound using induction. (Remember you may need to strengthen the inductive hypothesis from the obvious choice.)
- If you have extra time, you can consider proving asymptotic *lower bounds* for those same recurrences (e.g., prove the $\Omega(\cdot)$ bound by induction).

For review, the Master theorem considers recurrence relations of the following form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

for $a \geq 1$ and $b \geq 1$. The three cases state:

1) If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$.
2) If $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \lg n\right)$.
3) If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some constant $\epsilon > 0$ and if $a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Instructor's Examples

i) $T(n) = 2T\left(\frac{n}{2}\right) + n$. We briefly consider why the invalid inductive hypothesis that $T(n) \leq c \cdot n$ is fatally flawed.

ii) $T(n) = 2T\left(\frac{n}{2}\right) + n^2$. Solves to $T(n) = \Theta(n^2)$, using typical inductive hypothesis.

Inductive hypthesis: $T(n) \le c \cdot n^2$.
Base case: trivial for large enough c.
Inductive Case:

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + n^2 \\ &\le 2\left(c \cdot \left(\frac{n}{2}\right)^2\right) + n^2 = \frac{1}{2}c \cdot n^2 + n^2 \\ &\le c \cdot n^2 \qquad \text{for } c \ge 2 \end{aligned}$$

iii) $T(n) = 4T\left(\frac{n}{2}\right) + n$. Solves to $T(n) = \Theta(n^2)$, however the inductive hypothesis $T(n) \le c \cdot n^2$ fails. We strengthen hypothesis to $T(n) \le c \cdot n^2 - d \cdot n$ for some d.

Inductive hypthesis: $T(n) \le c \cdot n^2 - d \cdot n$.
Base case: trivial for large enough c.
Inductive Case:

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + n \\ &\le 4\left(c \cdot \left(\frac{n}{2}\right)^2 - d \cdot \frac{n}{2}\right) + n \\ &= c \cdot n^2 - 2d \cdot n + n = c \cdot n^2 - dn - n(d-1) \\ &\le c \cdot n^2 - d \cdot n \qquad \text{for } d \ge 1 \end{aligned}$$

Student Exercises

A) $T(n) = T\left(\frac{n}{2}\right) + 1$.

B) $T(n) = 2T\left(\frac{n}{2}\right) + 1$.

C) $T(n) = 4T\left(\frac{n}{3}\right) + n$.

D) $T(n) = 4T\left(\frac{n}{2}\right) + n^2$

E) $T(n) = 7T\left(\frac{n}{2}\right) + n^2$

F) $T(n) = 3T\left(\frac{n}{4}\right) + n \lg n$

G) $T(n) = T\left(\frac{7}{10}n\right) + T\left(\frac{1}{5}n\right) + n$

H) $T(n) = 2T\left(\sqrt{n}\right) + \lg n$. (Hint: consider substitution $m = \log n$.)

I) $T(n) = 3T\left(\sqrt{n}\right) + \lg n$.