CSCI 3100, Fall 2018

Extra credit [20 points] – greedy strategy proof

Consider the following problem from homework 3:

A small photocopying service business with a single large machine has the following scheduling problem. Each day they get a set of jobs from customers. They want to do the jobs on a single machine in an order that keeps customers happiest. Customer *i*'s job will take t_i time to complete. Given a schedule (an order of the jobs), let C_i denote the finishing time of job *i*. For example, if job *k* is the first to be done, we have $C_k=t_k$; and if job *k* is done right after job *i*, we have $C_k = C_i + t_k$. Each customer *i* also has a given weight w_i that represents his or her importance to the business. The happiness of customer *i* is expected to be dependent on the finishing time of job *i*. The company decides that they want to order the jobs to minimize the weighted sum of the completion times $\sum_{i=1}^{n} w_i C_i$.

In homework 3, you were asked to design an efficient algorithm to schedule the print jobs such that the weighted sum of completion times is minimized. In this extra credit assignment, I provide a greedy strategy for this problem. Your task is to prove that this greedy strategy leads to an optimal solution.

Type up your answer. I will not accept hand written solutions. Make sure your proof has a chain of logical statements. Hand waving proofs will receive no credit.

Greedy strategy: sort all jobs in descending order of w_i/t_i. This new order is the order in which print jobs should be schedule.

Prove that this greedy strategy leads to an optimal solution. You can use an 'exchange argument' for this. Start with: Suppose there is an optimal schedule O, where a pair of adjacently scheduled tasks x and y have: $w_x/t_x < w_y/t_y$ but task x is scheduled before task y. We can swap tasks x and y in schedule O to create schedule A. Now, we need to show that the weighted sum of schedule A is no bigger than the weighted sum of schedule O.

Write out the weighted sum of schedule O

Write out the weighted sum of schedule A

For jobs that are not x and y, what is those jobs' contribution to the weighted sum in schedule O? Did this contribution change in schedule A? Why?

Key factor to consider is this $w_x/t_x < w_y/t_y$ (from our greedy approach). This means that $w_x\, t_y < w_y\, t_x$

What is the contribution of job x to the weighted sum in schedule O? What is the contribution of job y to the weighted sum in schedule O? What is the total contribution of jobs x and y to the weighted sum in schedule O?

What is the contribution of job x to the weighted sum in schedule A? What is the contribution of job y to the weighted sum in schedule A? What is the total contribution of jobs x and y to the weighted sum in schedule A? How does this total compare to the total contribution of jobs x and y to the weighted sum in schedule O? What can we conclude from this?