CSCI 3100

MASTER THEOREM FOR RECURRENCES

Review and Overview

Previously, we solved recurrences with:

- Substitution method
- Recurrence trees/iteration

This time:

- Master theorem – recipe for some recurrences
- Does not apply to some cases

Master theorem

Given a recurrence: $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Compare f(n) to $n^{\log_b a}$

If f(n) is	Then T(n) is
$O(n^{\log_b a - \varepsilon})$	$\Theta(n^{\log_b a})$
$\Theta(n^{\log_b a})$	$\Theta(n^{\log_b a} * \lg(n))$
$\Omega(n^{\log_b a + \varepsilon})$	$\Theta(f(n))$

$$T(n) = 9T\left(\frac{n}{3}\right) + n$$

If f(n) is	Then T(n) is
$O(n^{\log_b a - \varepsilon})$	$\Theta(n^{\log_b a})$ ←
$\Theta(n^{\log_b a})$	$\Theta(n^{\log_b a} * \lg(n))$
$\Omega(n^{\log_b a + \varepsilon})$	$\Theta(f(n))$

$a = 9$, $b = 3$, $f(n) = n$ $\quad n^{\log_3 9} = n^2 \quad$ Let $\varepsilon = 0.1$

$n^{\log_b a - \varepsilon} = n^{1.9}$

$f(n) = n = O(n^{1.9}) \Rightarrow T(n) = \Theta(n^2)$

$$T(n) = T\left(\frac{2n}{3}\right) + 1$$

If f(n) is	Then T(n) is
$O(n^{log_b a - \varepsilon})$	$\Theta(n^{log_b a})$
$\Theta(n^{log_b a})$	$\Theta(n^{log_b a} * \lg(n))$
$\Omega(n^{log_b a + \varepsilon})$	$\Theta(f(n))$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$a = 1, \quad b = \frac{3}{2}, \quad f(n) = 1$

$n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$

$f(n) = \Theta(n^{\log_b a}) \Longrightarrow$ $T(n) = n^{\log_b a} * \lg n =$

$= \Theta(\lg(n))$

$$T(n) = 3T\left(\frac{n}{4}\right) + nlg(n)$$

If f(n) is	Then T(n) is
$O(n^{log_b a - \varepsilon})$	$\Theta(n^{log_b a})$
$\Theta(n^{log_b a})$	$\Theta(n^{log_b a} * \lg(n))$
$\Omega(n^{log_b a + \varepsilon})$	$\Theta(f(n))$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$\varepsilon < 1 - \log_4 3$

$a = 3, \quad b = 4, \quad f(n) = n \lg(n)$

$n^{\log_b a} = n^{\log_4 3}$ Let $\varepsilon = 0.1$

$n^{\log_b a + \varepsilon} = n^{\log_4 3 + 0.1}$ $= \Omega(n)$

$= \Omega(n \lg n) \Longrightarrow$

$T(n) = \Theta(n \lg n)$

$$T(n) = 2T\left(\frac{n}{2}\right) + nlg(n)$$

If f(n) is	Then T(n) is
$O(n^{log_b a - \varepsilon})$	$\Theta(n^{log_b a})$
$\Theta(n^{log_b a})$	$\Theta(n^{log_b a} * \lg(n))$
$\Omega(n^{log_b a + \varepsilon})$	$\Theta(f(n))$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$a = 2,\ b = 2,\ f(n) = n\lg(n)$

$n^{\log_b a} = n^{\log_2 2} = n^{(1)}$

$n\lg(n) \neq \Omega(n^{1+\varepsilon})$

$n^{1+\varepsilon} = n \cdot n^{\varepsilon}$ $\qquad \frac{f(n)}{n^{\log_b a + \varepsilon}} = \frac{\cancel{n}\lg n}{\cancel{n} \cdot n^{\varepsilon}} < 1 \quad n \to \infty$

Master theorem does <u>NOT</u> apply

$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

If f(n) is	Then T(n) is
$O(n^{log_b a - \varepsilon})$	$\Theta(n^{log_b a})$
$\Theta(n^{log_b a})$	$\Theta(n^{log_b a} * \lg(n))$ ←
$\Omega(n^{log_b a + \varepsilon})$	$\Theta(f(n))$

$a = 2,\ b = 2,\ f(n) = \Theta(n)$

$n^{\log_b a} = n^{\log_2 2} = n$

$f(n) = \Theta(n) \implies T(n) = \Theta(n \cdot \lg(n))$

$T(n) = 8T(\frac{n}{2}) + \Theta(n^2)$

$T(n) = aT(\frac{n}{b}) + f(n)$

If f(n) is	Then T(n) is
$O(n^{\log_b a - \varepsilon})$	$\Theta(n^{\log_b a})$
$\Theta(n^{\log_b a})$	$\Theta(n^{\log_b a} * \lg(n))$
$\Omega(n^{\log_b a + \varepsilon})$	$\Theta(f(n))$

$a = 8, \; b = 2, \; f(n) = \Theta(n^2)$

$n^{\log_b a} = n^{\log_2 8} = n^3$

$\Theta(n^2) = O(n^{3-\varepsilon}) \qquad \varepsilon = 1$

$O(n^2) \Rightarrow T(n) = \Theta(n^3)$

$T(n) = 7T(\frac{n}{2}) + \Theta(n^2)$

If f(n) is	Then T(n) is
$O(n^{\log_b a - \varepsilon})$	$\Theta(n^{\log_b a})$
$\Theta(n^{\log_b a})$	$\Theta(n^{\log_b a} * \lg(n))$
$\Omega(n^{\log_b a + \varepsilon})$	$\Theta(f(n))$

$T(n) = aT(\frac{n}{b}) + f(n)$

$a = 7, \; b = 2, \; \underline{f(n) =} \Theta(n^2)$

$n^{\log_b a} = n^{\log_2 7} \approx \underline{n^{2.8}} = n^2 \cdot n^{0.8} \qquad$ Let $\varepsilon = 0.8$

$n^{\log_b a} \not\approx n^2$

$f(n) = \Theta(n^2) = O(n^{\log_b a - \varepsilon}) \Rightarrow T(n) \approx \Theta(n^{2.8})$