Elementary Graph Algorithms

CSCI 3100

Graphs

Graph $G = (V, E)$

- V = set of vertices
- E = set of edges $\subseteq (V \times V)$

Types of graphs

- Undirected: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
- Directed: (u, v) is edge from u to v, denoted as $u \rightarrow v$. Self loops are allowed.
- Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
- Dense: $|E| \approx |V|^2$.
- Sparse: $|E| << |V|^2$.

$|E| = O(|V|^2)$

Graphs

If $(u, v) \in E$, then vertex v is adjacent to vertex u.

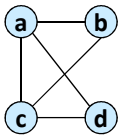
Adjacency relationship is:
- Symmetric if G is undirected.
- Not necessarily so if G is directed.

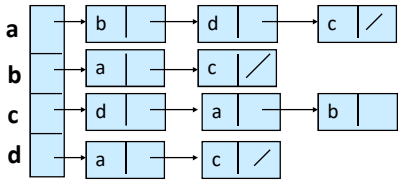
If G is connected:
- There is a path between every pair of vertices.
- $|E| \geq |V| - 1$.
- Furthermore, if $|E| = |V| - 1$, then G is a tree.

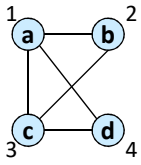
Other definitions in Appendix B (B.4 and B.5) as needed.

Representation of Graphs

Two standard ways.
- Adjacency Lists.
- Adjacency Matrix.







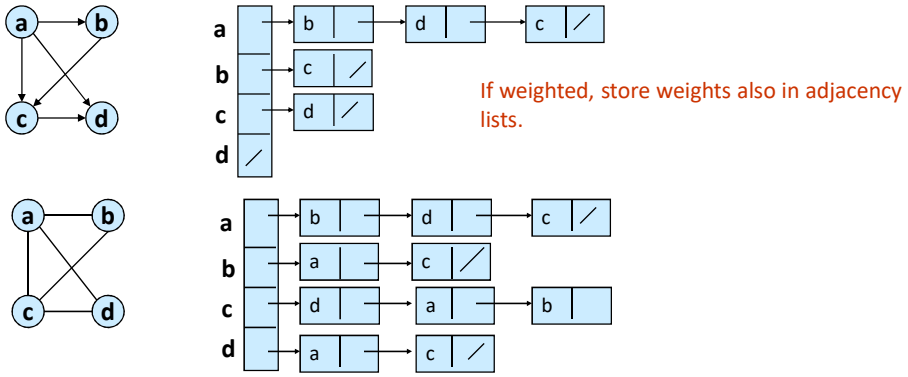
	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

Adjacency Lists

Consists of an array *Adj* of $|V|$ lists.

One list per vertex.

For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u.



Storage Requirement

For directed graphs:

- Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{out-degree}(v) = |E|$$

No. of edges leaving v

- Total storage: $\Theta(V+E)$

For undirected graphs:

- Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

No. of edges incident on v. Edge (u,v) is incident on vertices u and v.

- Total storage: $\Theta(V+E)$

Pros and Cons: adjacency list

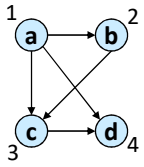
Pros

- Space-efficient, when a graph is sparse.
- Can be modified to support many graph variants.

Cons

- Determining if an edge $(u,v) \in G$ is not efficient.
  - Have to search in u's adjacency list. Time complexity: $\Theta(degree(u))$
  - Worst case time complexity to search u's adjacency list: $\Theta(|V|)$

Adjacency Matrix



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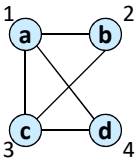
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$|V| \times |V|$ matrix A.

Number vertices from 1 to $|V|$ in some arbitrary manner.

A is then given by:

$$A[i,j] = a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

$A = A^T$ for undirected graphs.

Space and Time

Space: $\Theta(|V|^2)$.

- Not memory efficient for large graphs.

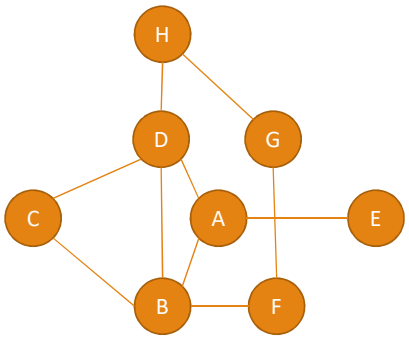
Time: to list all vertices adjacent to u: $\Theta(|V|)$.

Time: to determine if $(u, v) \in E$: $\Theta(1)$.

Can store weights instead of bits for weighted graph.

Exercise

Draw adjacency list and adjacency matrix for the following graph:



Graph-searching Algorithms

Searching a graph:

- Systematically follow the edges of a graph to visit the vertices of the graph.

Used to discover the structure of a graph.

Standard graph-searching algorithms.

- Breadth-first Search (BFS).
- Depth-first Search (DFS).

Breadth-first Search

Input: Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.

Output:

- $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
- $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \rightsquigarrow v$.
  - u is v's predecessor.
- Builds breadth-first tree with root s that contains all reachable vertices.

Definitions:
Path between vertices u and v: Sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $u=v_1$ and $v=v_k$, and $(v_i, v_{i+1}) \in E$, for all $1 \le i \le k-1$.
Length of the path: Number of edges in the path.
Path is simple if no vertex is repeated.

Breadth-first Search

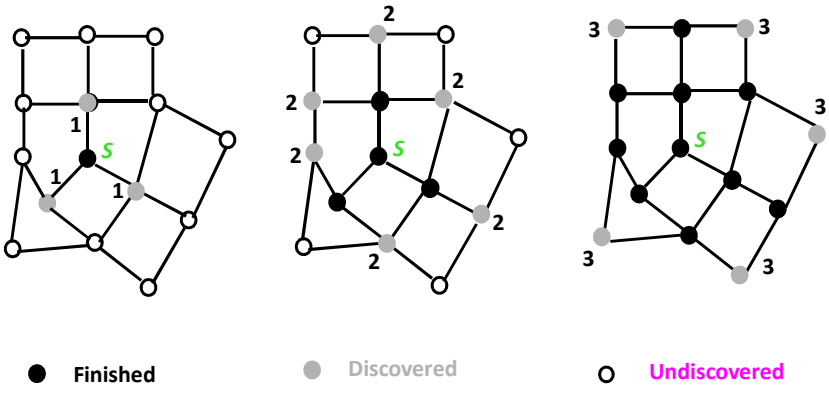
Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.

- A vertex is "discovered" the first time it is encountered during the search.
- A vertex is "finished" if all vertices adjacent to it have been discovered.

Colors the vertices to keep track of progress.

- White – Undiscovered.
- Gray – Discovered but not finished.
- Black – Finished.
  - Colors are required only to reason about the algorithm. Can be implemented without colors.

BFS for Shortest Paths

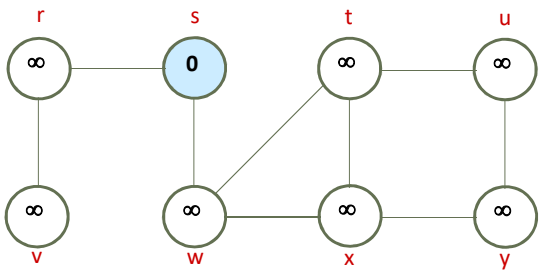


```
BFS(G, s)
 1  for each vertex u ∈ G.V − {s}
 2      u.color = WHITE
 3      u.d = ∞
 4      u.π = NIL
 5  s.color = GRAY
 6  s.d = 0
 7  s.π = NIL
 8  Q = ∅
 9  ENQUEUE(Q, s)
10  while Q ≠ ∅
11      u = DEQUEUE(Q)
12      for each v ∈ G.Adj[u]
13          if v.color == WHITE
14              v.color = GRAY
15              v.d = u.d + 1
16              v.π = u
17              ENQUEUE(Q, v)
18      u.color = BLACK
```

white: undiscovered
gray: discovered
black: finished

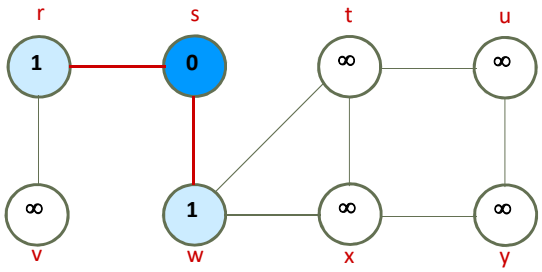
Q: a queue of discovered vertices
v.color: color of v
v.d: distance from s to v
v.π: predecessor of v

Example (BFS)



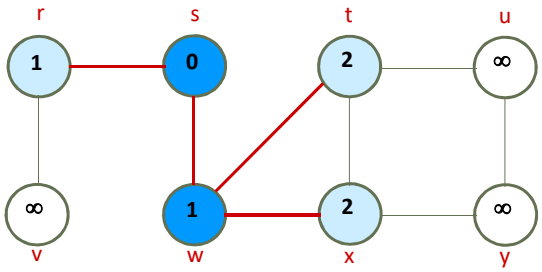
Q: s
0

Example (BFS)



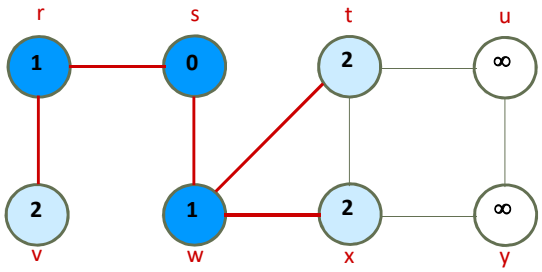
Q: w r
1 1

Example (BFS)



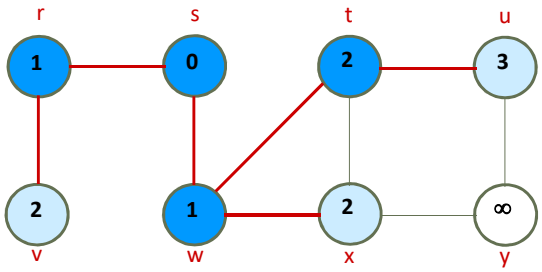
Q: r t x
1 2 2

Example (BFS)



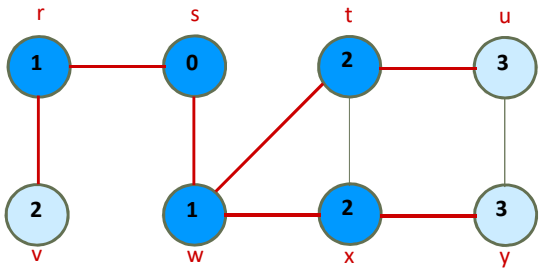
Q: t x v
2 2 2

Example (BFS)



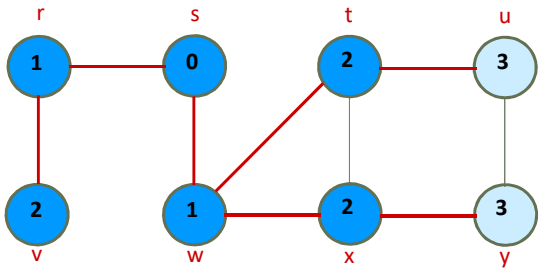
Q: x v u
2 2 3

Example (BFS)



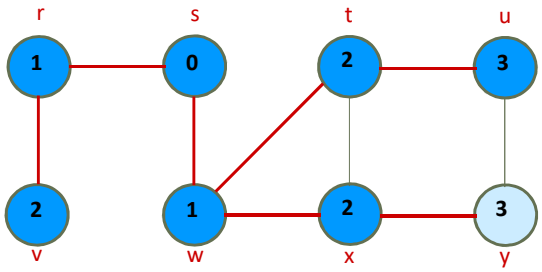
Q: v u y
2 3 3

Example (BFS)



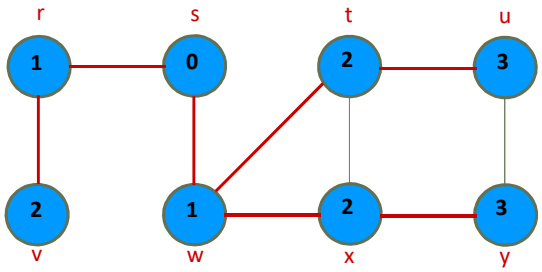
Q: u y
3 3

Example (BFS)



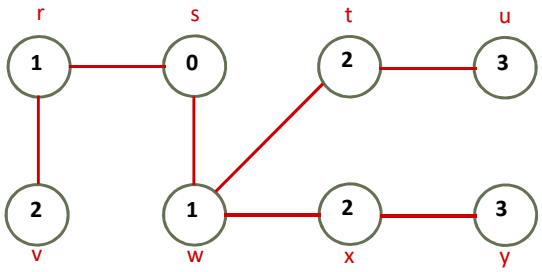
Q: y
3

Example (BFS)



Q: ∅

Example (BFS)



BF Tree

Analysis of BFS

Initialization takes $O(V)$.

Traversal Loop

- After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(V)$.
- The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(E)$.

Summing up over all vertices => total running time of BFS is $O(V+E)$, linear in the size of the adjacency list representation of graph.

```
BFS(G, s)
 1  for each vertex u ∈ G.V − {s}
 2      u.color = WHITE
 3      u.d = ∞
 4      u.π = NIL
 5  s.color = GRAY
 6  s.d = 0
 7  s.π = NIL
 8  Q = ∅
 9  ENQUEUE(Q, s)
10  while Q ≠ ∅
11      u = DEQUEUE(Q)
12      for each v ∈ G.Adj[u]
13          if v.color == WHITE
14              v.color = GRAY
15              v.d = u.d + 1
16              v.π = u
17              ENQUEUE(Q, v)
18      u.color = BLACK
```

Breadth-first Tree

For a graph $G = (V, E)$ with source s, the **predecessor subgraph** of G is $G_\pi = (V_\pi, E_\pi)$ where

- $V_\pi = \{v \in V : \pi[v] \neq \text{NIL}\} \cup \{s\}$
- $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$

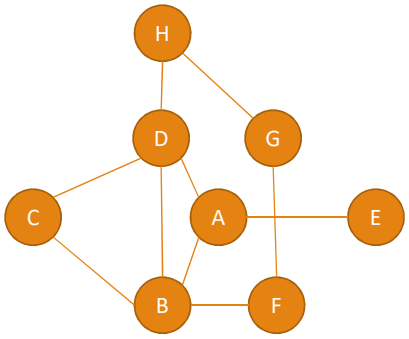
The predecessor subgraph G_π is a **breadth-first tree** if:

- V_π consists of the vertices reachable from s and
- for all $v \in V_\pi$, there is a unique simple path from s to v in G_π that is also a shortest path from s to v in G.

The edges in E_π are called **tree edges**. $|E_\pi| = |V_\pi| - 1$.

Exercise

Use BFS algorithm to determine the breadth-first tree of the following graph starting with node A:



Shortest Path

Breadth First Search computes the shortest path between two nodes of a graph

Proof is in Ch 22.2 – READ IT 

Depth-first Search (DFS)

Explore edges out of the most recently discovered vertex *v*.

When all edges of *v* have been explored, backtrack to explore other edges leaving the vertex from which *v* was discovered (its *predecessor*).

"Search as deep as possible first."

Continue until all vertices reachable from the original source are discovered.

If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.

Depth-first Search

Input: $G = (V, E)$, directed or undirected. No source vertex given!

Output:

- 2 **timestamps** on each vertex. Integers between 1 and 2|V|.
  - $d[v]$ = ***discovery time*** (v turns from white to gray)
  - $f[v]$ = ***finishing time*** (v turns from gray to black)
- $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.

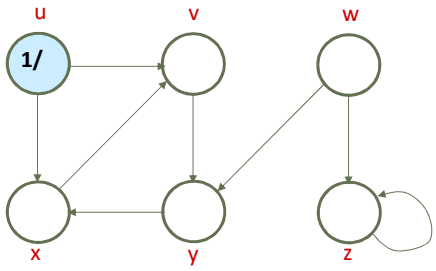
Uses the same coloring scheme for vertices as BFS.

Pseudo-code

```
DFS(G)
1  for each vertex u ∈ G.V
2      u.color = WHITE
3      u.π = NIL
4  time = 0
5  for each vertex u ∈ G.V
6      if u.color == WHITE
7          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
1  time = time + 1
2  u.d = time
3  u.color = GRAY
4  for each v ∈ G.Adj[u]
5      if v.color == WHITE
6          v.π = u
7          DFS-VISIT(G, v)
8  u.color = BLACK
9  time = time + 1
10 u.f = time
```

Example (DFS)



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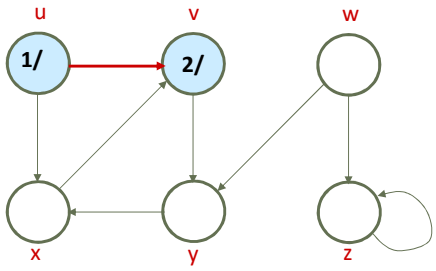

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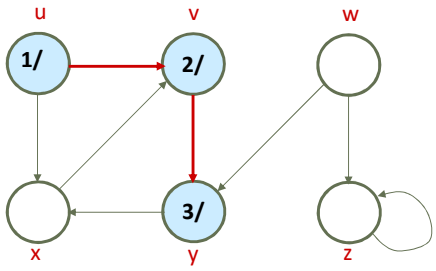
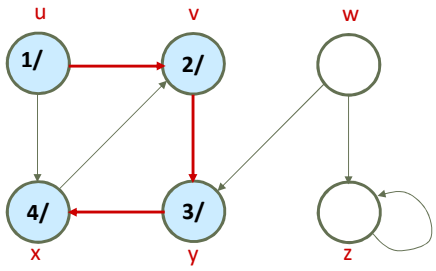


Example (DFS)



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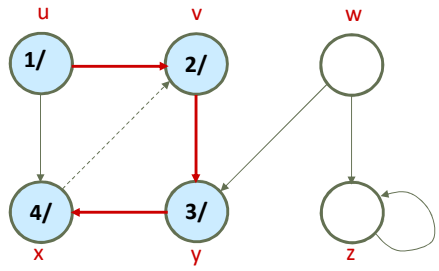

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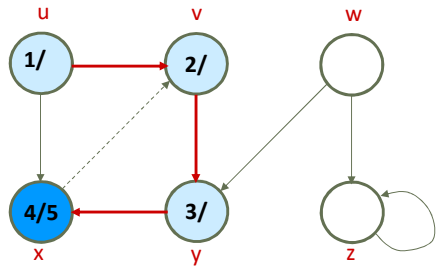

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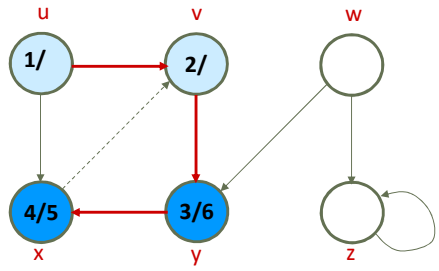

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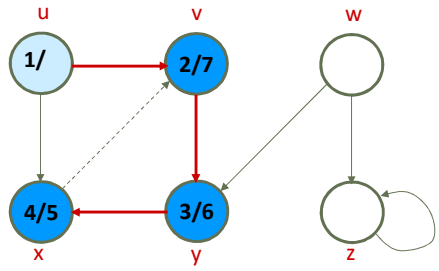

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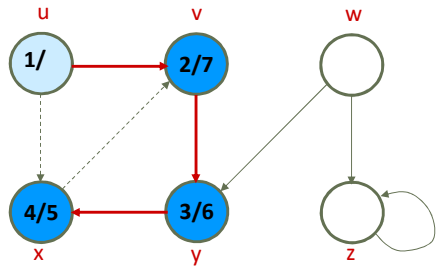

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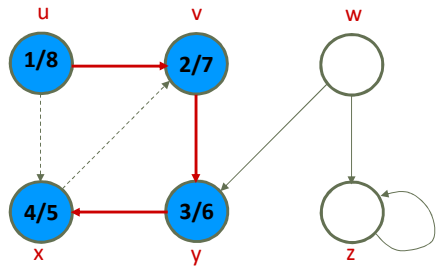

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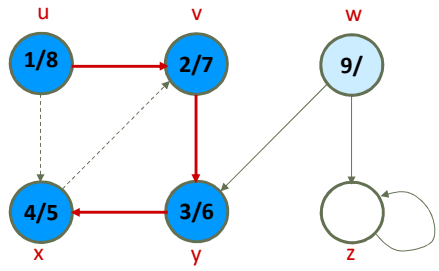

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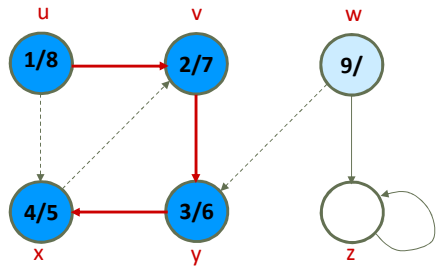

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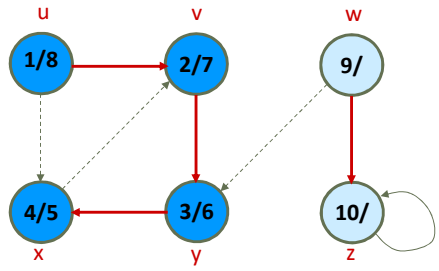

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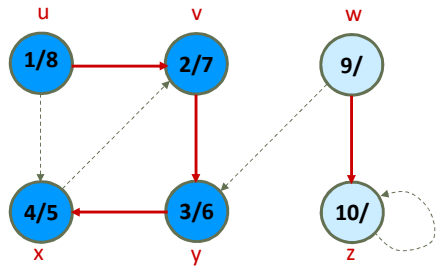

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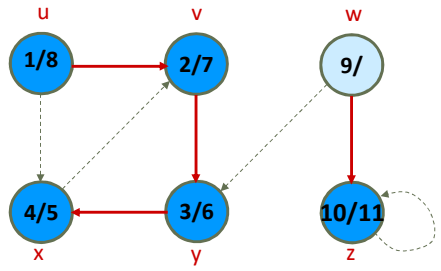

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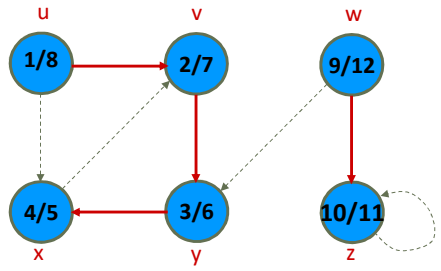


Analysis of DFS

Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.

DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 4-7 of DFS-Visit is executed $|Adj[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |Adj[v]| = \Theta(E)$

Total running time of DFS is $\Theta(V+E)$.

```
DFS(G)
1  for each vertex u ∈ G.V
2      u.color = WHITE
3      u.π = NIL
4  time = 0
5  for each vertex u ∈ G.V
6      if u.color == WHITE
7          DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
1  time = time + 1
2  u.d = time
3  u.color = GRAY
4  for each v ∈ G.Adj[u]
5      if v.color == WHITE
6          v.π = u
7          DFS-VISIT(G, v)
8  u.color = BLACK
9  time = time + 1
10 u.f = time
```

Theorem 22.7

For all u, v, exactly one of the following holds:

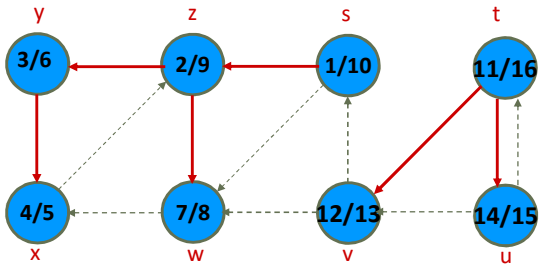
1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither u nor v is a descendant of the other.
2. $d[u] < d[v] < f[v] < f[u]$ and v is a descendant of u.
3. $d[v] < d[u] < f[u] < f[v]$ and u is a descendant of v.

- So $d[u] < d[v] < f[u] < f[v]$ *cannot* happen.
- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])

Corollary

v is a proper descendant of u if and only if $d[u] < d[v] < f[v] < f[u]$.

Example (Parenthesis Theorem)



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

Depth-First Trees

Predecessor subgraph defined slightly different from that of BFS.

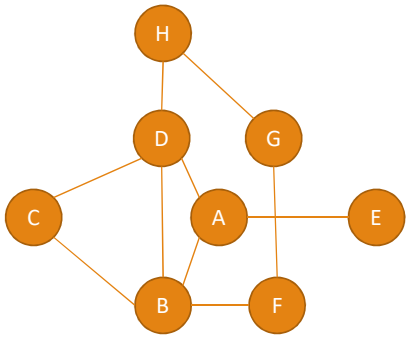
The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq \text{NIL}\}$.

- How does it differ from that of BFS?
- The predecessor subgraph G_π forms a *depth-first forest* composed of several *depth-first trees*. The edges in E_π are called *tree edges*.

Definition:
Forest: An acyclic graph G that may be disconnected.

Exercise

Apply DFS algorithm to the following graph and draw the depth first forest.



Theorem 22.9

v is a descendant of *u* if and only if at time *d*[*u*], there is a path from *u to v* consisting of only white vertices. (Except for *u*, which was *just* colored gray.)

Classification of Edges

Tree edge: in the depth-first forest. Found by exploring (u, v).

Back edge: (u, v), where u is a descendant of v (in the depth-first tree).

Forward edge: (u, v), where v is a descendant of u, but not a tree edge.

Cross edge: any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

Theorem:
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.