Graph Algorithms

Topological Sort
Strongly Connected Components

CSCI 3100

Review & Overview

Last week:

- Introduced the concept of Graph
- Graph terminology
- Graph representation
- Graph traversal algorithms:
  - Breadth first search
  - Depth first search

This week:

- Continue with graph algorithms
- Topological sort of vertices
- Strongly connected components
- Midterm review - Friday

Identification of Edges

Edge type for edge (u, v) can be identified when it is first explored by DFS.

Identification is based on the **color of *v***.

- White – tree edge.
- Gray – back edge.
- Black – forward or cross edge.

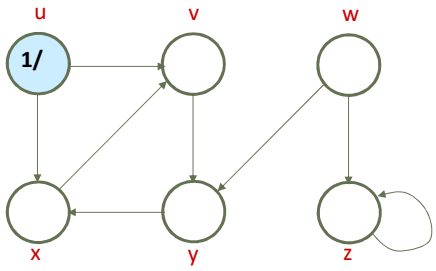
Tree edge: in the depth-first forest. Found by exploring (u, v).

Back edge: (u, v), where u is a descendant of v (in the depth-first tree).

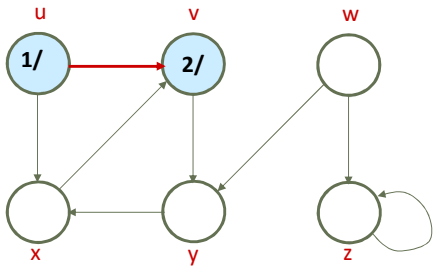
Forward edge: (u, v), where v is a descendant of u, but not a tree edge.

Cross edge: any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

Example (DFS)



Example (DFS)



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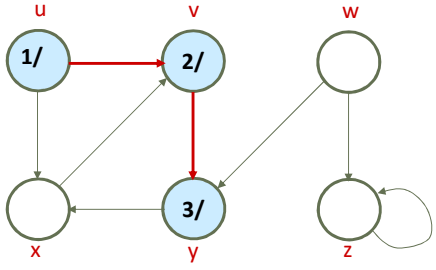

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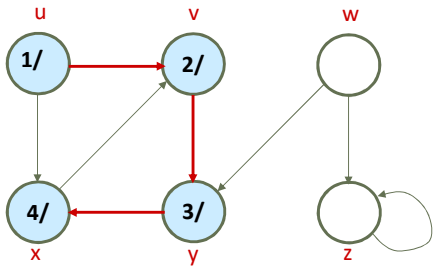

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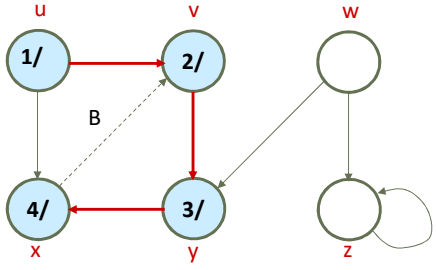

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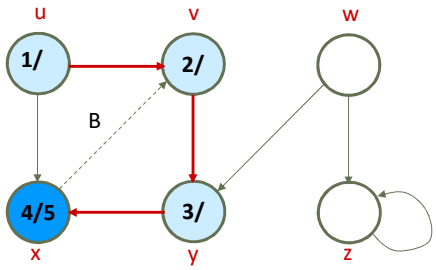

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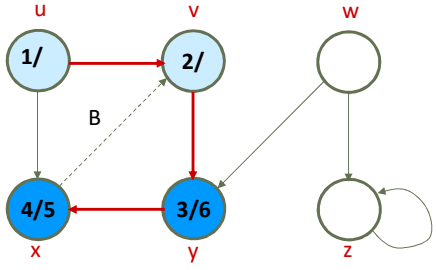

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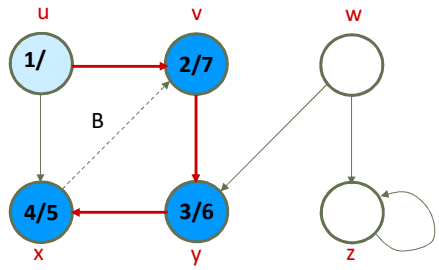

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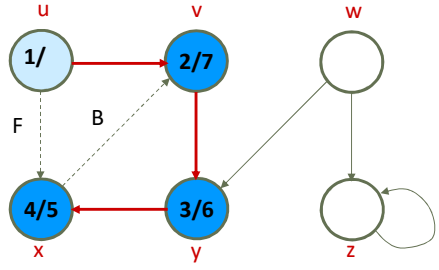

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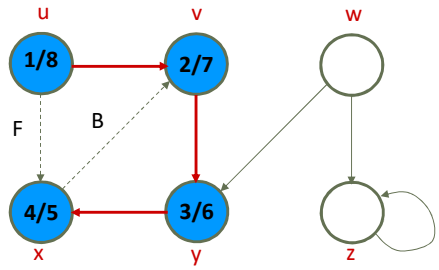

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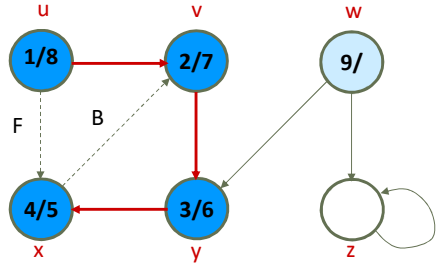

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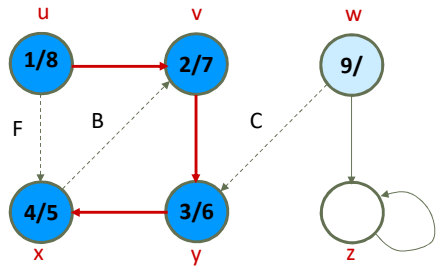

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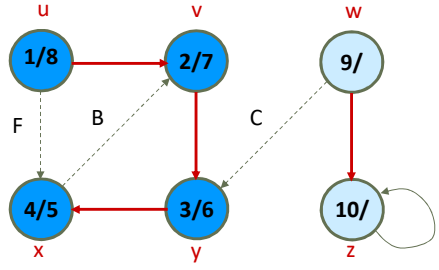

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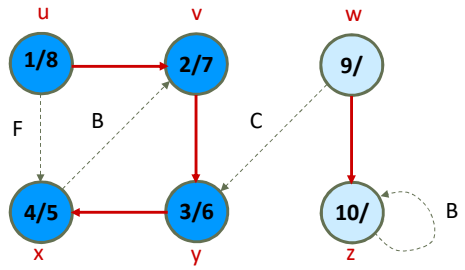

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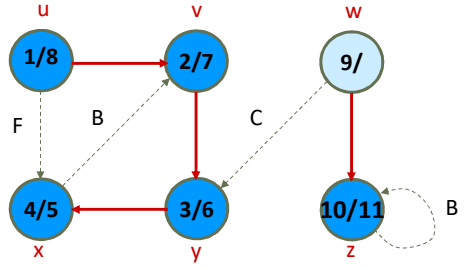

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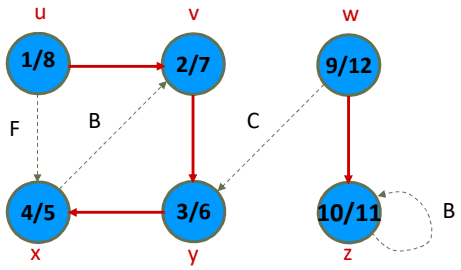


Which is correct for DFS?

Recall:

- d[u] – discovery time of vertex u
- f[u] – finish time of vertex u

Given a directed edge (u, v) in graph G, the following is possible:

A. d[u] < d[v] < f[v] < f[u]

B. d[v] < d[u] < f[u] < f[v]

C. d[u] < f[u] < d[v] < f[v]

D. d[u] < d[v] < f[u] < f[v]

Directed Acyclic Graph

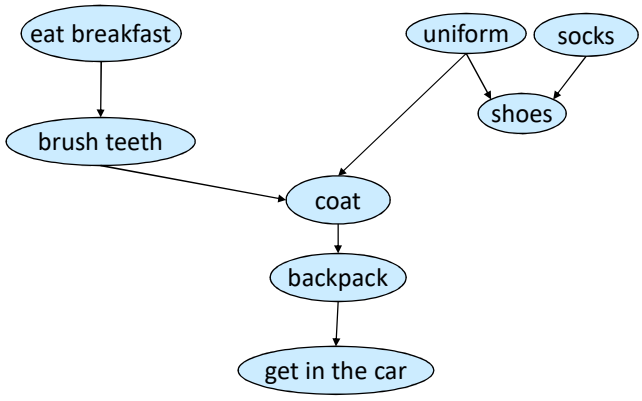
DAG – Directed graph with no cycles.

Good for modeling processes and structures that have a **partial order:**

- $a > b$ and $b > c \Rightarrow a > c$.
- But may have a and b such that neither $a > b$ nor $b > a$.

Can always make a **total order** (either $a > b$ or $b > a$ for all $a \neq b$) from a partial order.

Example

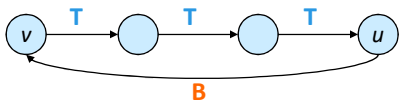


Lemma 22.11
A directed graph *G* is acyclic iff a DFS of G yields no back edges.

Proof:

⇒: Show that back edge ⇒ cycle.

- Suppose there is a back edge (u, v). Then v is ancestor of u in depth-first forest.
- Therefore, there is a path from V to u, so $v \rightsquigarrow u \rightsquigarrow v$ is a cycle.



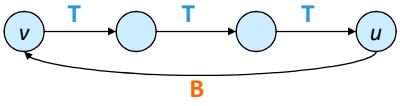
Characterizing a DAG

Lemma 22.11
A directed graph *G* is acyclic iff a DFS of G yields no back edges.

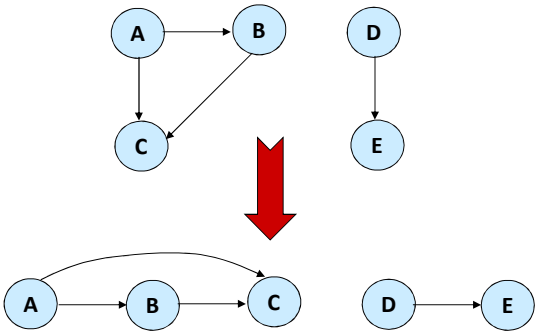
Proof (Contd.):

⇐ : Show that a cycle implies a back edge.

- c : cycle in G, v : first vertex discovered in c, (u, v) : preceding edge in c.
- At time $d[v]$, vertices of c form a white path $v \rightsquigarrow u$. Why?
- u is a descendent of v in depth-first forest.
- Therefore, (u, v) is a back edge.



Topological Sort: "sort" DAG



Think of original DAG as a **partial order**.

Want a **total order** that extends this partial order.

Topological Sort

Performed on a DAG.

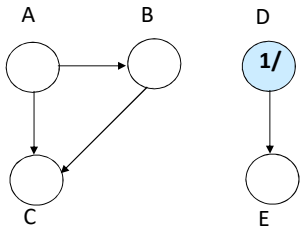
Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears somewhere before v.

Topological-Sort (G)

1. call DFS(G) to compute finishing times $f[v]$ for all $v \in V$
2. as each vertex is finished, insert it onto the front of a linked list
3. **return** the linked list of vertices

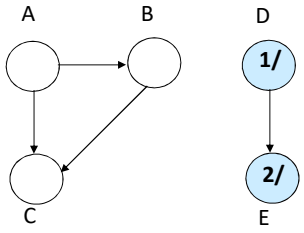
Time: $\Theta(V + E)$.

Example



Linked List:

Example



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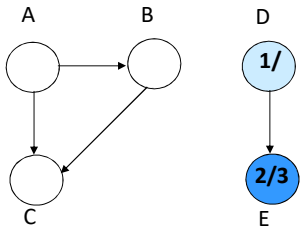

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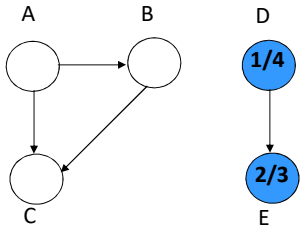

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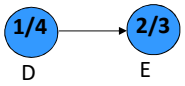

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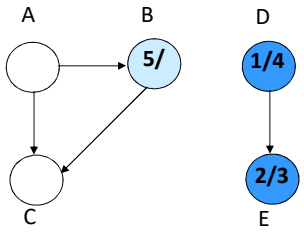

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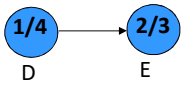

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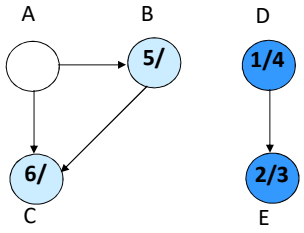

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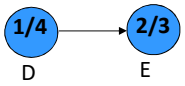

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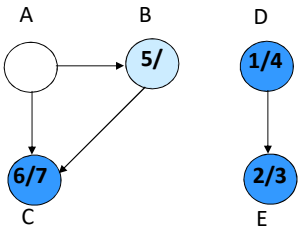

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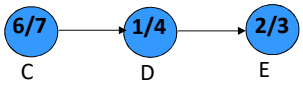

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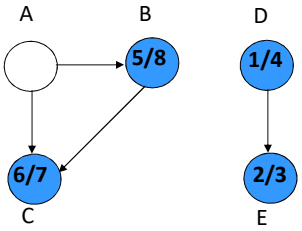

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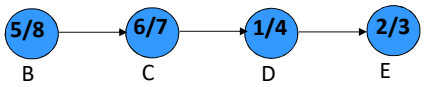

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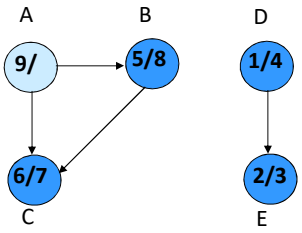

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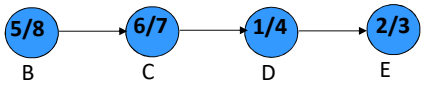

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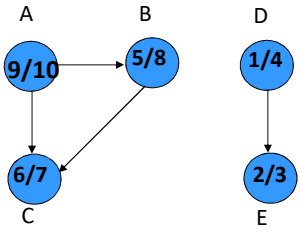

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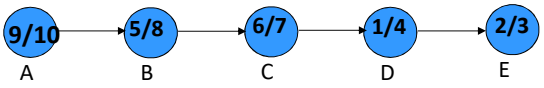
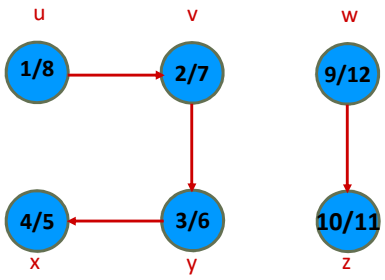


What is the correct topological sort order of this DFS forest?



A. u->v->y->x->w->z
B. z->w->x->y->v->u
C. w->z->u->v->y->x
D. z->x->v->y->w->z

Correctness Proof

Just need to show if $(u, v) \in E$, then $f[v] < f[u]$.

When we explore (u, v), what are the colors of u and v?

- u is gray.
- Is v gray, too?
  - *No*, because then v would be ancestor of u.
  - $\Rightarrow (u, v)$ is a back edge.
  - $\Rightarrow$ contradiction of Lemma 22.11 (dag has no back edges).
- Is v white?
  - Then becomes descendant of u.
  - By parenthesis theorem, $d[u] < d[v] < \underline{f[v] < f[u]}$.
- Is v black?
  - Then v is already finished.
  - Since we're exploring (u, v), we have not yet finished u.
  - Therefore, $f[v] < f[u]$.

Can we adjust Breadth First Search algorithm to do topological ordering of a DAG

A. No, because in Breadth First search there is no guarantee that if d[u]<d[v] then f[v] < f[u]

B. No, because Breadth First Search finishes exploring a given vertex before exploring gray vertices

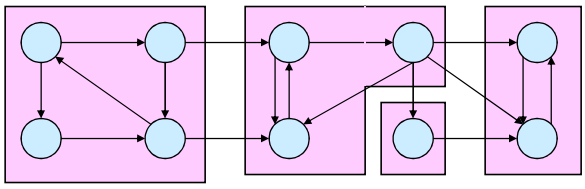
C. Yes, if we start with a vertex that has 0 in-degree

D. Yes, if the DAG is connected

Strongly Connected Components

G is strongly connected if every pair (*u*, *v*) of vertices in *G* is reachable from one another.

A **strongly connected component** (***SCC***) of *G* is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \rightsquigarrow v$ and $v \rightsquigarrow u$ exist.



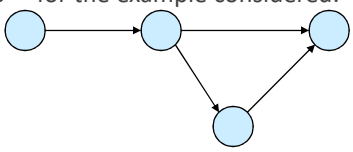
Component Graph

$G^{SCC} = (V^{SCC}, E^{SCC})$.

V^{SCC} has one vertex for each SCC in G.

E^{SCC} has an edge if there's an edge between the corresponding SCC's in G.

G^{SCC} for the example considered:



G^{SCC} is a DAG

Lemma 22.13

Let C and C' be distinct SCC's in G, let $u, v \in C$, $u', v' \in C'$, and suppose there is a path $u \rightsquigarrow u'$ in G. Then there cannot also be a path $v' \rightsquigarrow v$ in G.

Proof:

Suppose there is a path $v' \rightsquigarrow v$ in G.

Then there are paths $u \rightsquigarrow u' \rightsquigarrow v'$ and $v' \rightsquigarrow v \rightsquigarrow u$ in G.

Therefore, u and v' are reachable from each other, so they are not in separate SCC's.

Transpose of a Directed Graph

G^T = **transpose** of directed G.

- $G^T = (V, E^T)$, $E^T = \{(u, v) : (v, u) \in E\}$.
- G^T is G with all edges reversed.

Can create G^T in $\Theta(V + E)$ time if using adjacency lists.

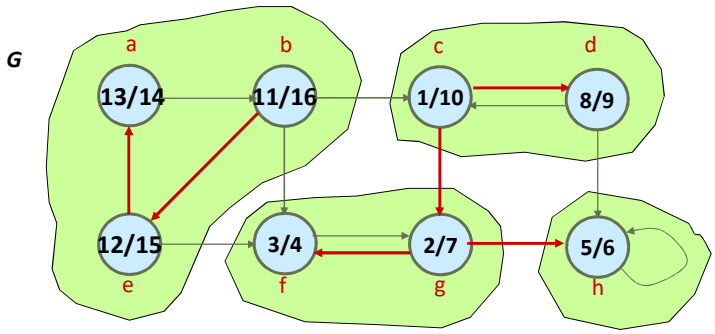
G and G^T have the *same* SCC's. (u and v are reachable from each other in G if and only if reachable from each other in G^T.)

SCC(G)

1. call DFS(G) to compute finishing times $f[u]$ for all u
2. compute G^T
3. call DFS(G^T), but in the main loop, consider vertices in order of decreasing $f[u]$ (as computed in first DFS)
4. output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

Time: $\Theta(V + E)$.

Example



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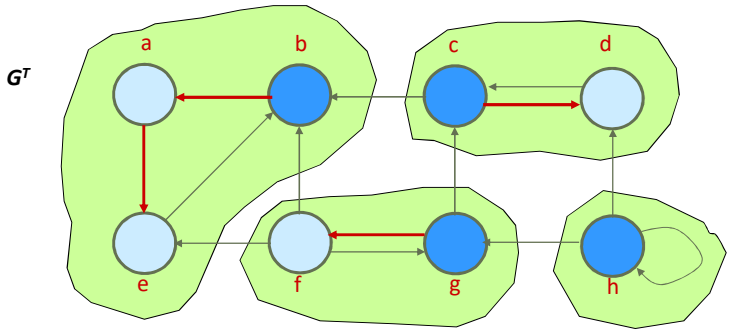

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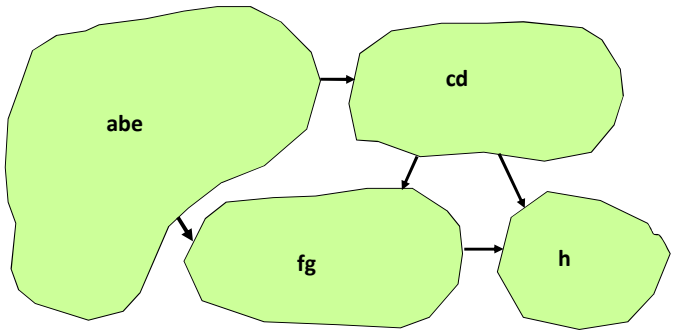


How does it work?

Idea:

- By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the component graph in topologically sorted order.
- Because we are running DFS on G^T, we will not be visiting any v from u, where v and u are in different components.

Notation:

- $d[u]$ and $f[u]$ always refer to *first* DFS.
- Extend notation for d and f to sets of vertices $U \subseteq V$:
- $d(U) = \min_{u \in U}\{d[u]\}$ (earliest discovery time)
- $f(U) = \max_{u \in U}\{f[u]\}$ (latest finishing time)

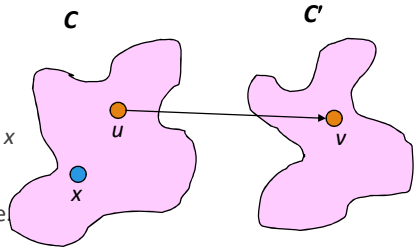
SCCs and DFS finishing times

Lemma 22.14
Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Proof:

Case 1: $d(C) < d(C')$

- Let x be the first vertex discovered in C.
- At time $d[x]$, all vertices in C and C' are white. Thus, there exist paths of white vertices from x to all vertices in C and C'.
- By the white-path theorem, all vertices in C and C' are descendants of x in depth-first tree.
- By the parenthesis theorem, $f[x] = f(C) > f(C')$.



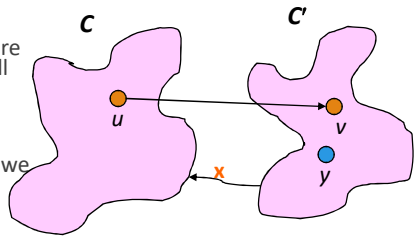
SCCs and DFS finishing times

Lemma 22.14
Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Proof:

Case 2: $d(C) > d(C')$

- Let y be the first vertex discovered in C'.
- At time $d[y]$, all vertices in C' are white and there is a white path from y to each vertex in $C' \Rightarrow$ all vertices in C' become descendants of y. Again, $f[y] = f(C')$.
- At time $d[y]$, all vertices in C are also white.
- By earlier lemma, since there is an edge (u, v), we cannot have a path from C' to C.
- So no vertex in C is reachable from y.
- Therefore, at time $f[y]$, all vertices in C are still white.
- Therefore, for all $w \in C$, $f[w] > f[y]$, which implies that $f(C) > f(C')$.



SCCs and DFS finishing times

Corollary 22.15
Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E^T$, where $u \in C$ and $v \in C'$. Then $f(C) < f(C')$.

Proof:

$(u, v) \in E^T \Rightarrow (v, u) \in E$.

Since SCC's of G and G^T are the same, $f(C') > f(C)$, by Lemma 22.14.

Correctness of SCC

When we do the second DFS, on G^T, start with SCC C such that $f(C)$ is maximum.

- The second DFS starts from some $x \in C$, and it visits all vertices in C.
- Corollary 22.15 says that since $f(C) > f(C')$ for all $C \neq C'$, there are no edges from C to C' in G^T.
- Therefore, DFS will visit *only* vertices in C.
- Which means that the depth-first tree rooted at x contains *exactly* the vertices of C.

Correctness of SCC

The next root chosen in the second DFS is in SCC C' such that $f(C')$ is maximum over all SCC's other than C.

- DFS visits all vertices in C', but the only edges out of C' go to C, *which we've already visited*.
- Therefore, the only tree edges will be to vertices in C'.

We can continue the process.

Each time we choose a root for the second DFS, it can reach only

- vertices in its SCC—get tree edges to these,
- vertices in SCC's *already visited* in second DFS—get *no* tree edges to these.