Max Flow & Min Cut

CSCI 3100

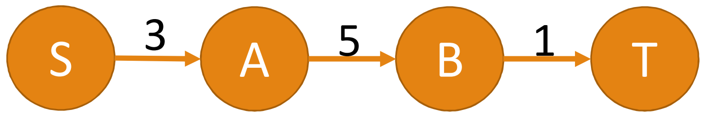
Augmenting Path

Given a network G = (V, E) and a flow f, **augmenting path** *p* is a simple path from s to t in the residual network G_f.

We can increase the flow along the augmenting path by up to $c_f(u,v)$ without violating the capacity constraint of (u, v) or (v, u) (whichever edge was in the original network).

Residual capacity of an augmenting path p is the capacity of the smallest edge along p. We'll refer to it as **$c_f(p)$.**

What is the **residual capacity** $c_f(p)$ of path p=<S, A, B, T> below?



A. 3

B. 5

C. 1

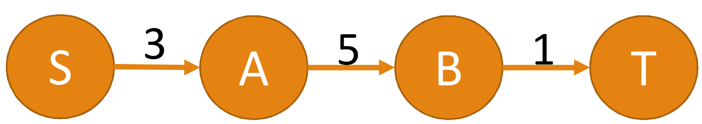
D. 9

E. None of the above

Flow along augmenting path

$$f_p(x, y) = \begin{cases} c_f(p) & if\ (x, y)\ is\ on\ p \\ 0 & otherwise \end{cases}$$

What is the flow f_p along the augmenting path <S, A, B, T> below (edge weights are capacities)?



	f_p(S, A)	f_p(A, B)	f_p(B, T)
A.	1	1	1
B.	3	5	1
C.	5	5	5

$|f \uparrow f_p| = |f| + |f_p| > |f|$

Last time we showed that $|f \uparrow f'| = |f| + |f'|$

We know that $|f_p| > 0$

Ford-Fulkerson Method Revisited

Repeatedly augment the flow along augmenting paths until we find the maximum flow.

How do we know that we found the maximum flow?

Max Flow Min Cut Theorem:

Flow is maximum if and only if its residual network contains no augmenting paths.

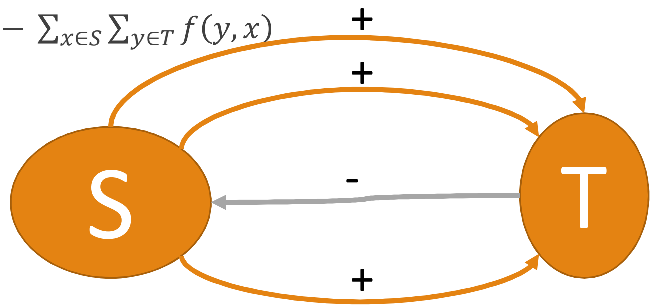
To prove the theorem, we need to introduce the concept of a “cut”

Cut of a flow network

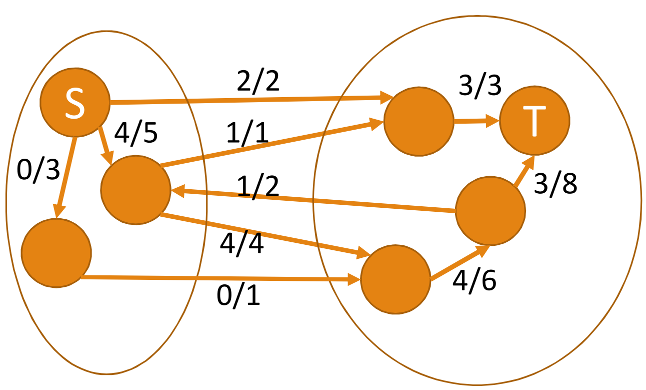
A **cut** *(S, T)* of a flow network *G=(V, E)* is a partition of *V* into two subsets *S* and *T=V-S* such that $s \in S$ and $t \in T$.

A **net flow *f(S, T)*** across a cut *(S, T)* is defined as

$$f(S,T) = \sum_{x\in S}\sum_{y\in T} f(x,y) - \sum_{x\in S}\sum_{y\in T} f(y,x)$$



What is the flow across the cut *(S, T)*? The edge labels show flow/capacity.



A. 8

B. 7

C. 6

D. 10

E. 1

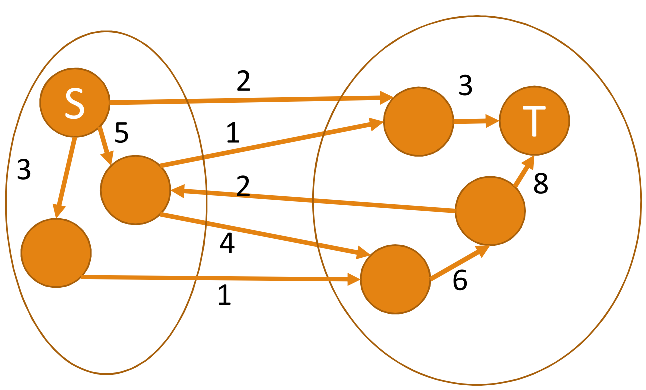
Capacity of a cut *(S, T)*

The sum of capacities of edges going from S to T

$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$

Minimum Cut – cut *(S, T)* of a network with the smallest capacity (over all *(S, T)* cuts of a network).

What is the **capacity** of the cut *(S, T)*? The edge labels show edge capacity.



A. 8

B. 7

C. 6

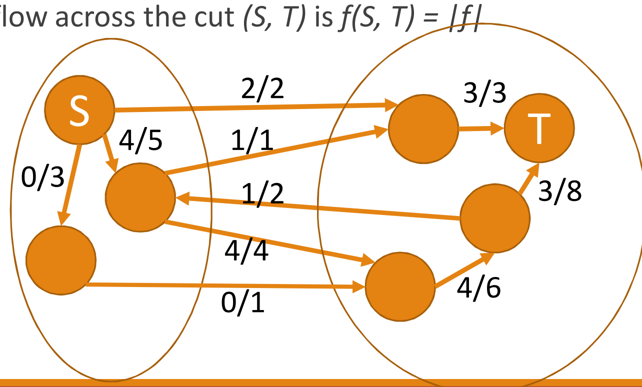
D. 10

E. 1

Flow across a cut

Lemma 26.4

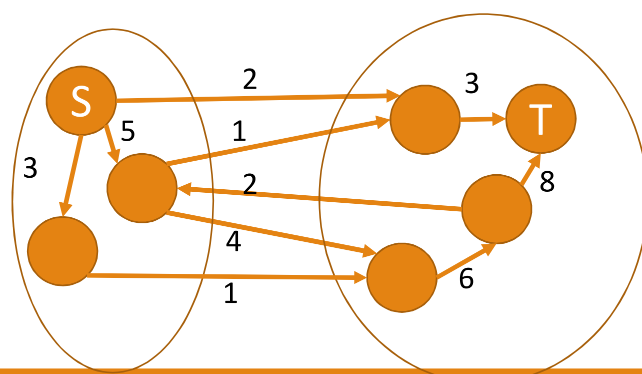
Let *f* be a flow in a network *G=(V, E)* with source *s* and sink *t*, and let *(S, T)* be any cut of *G*. Then the flow across the cut *(S, T)* is *f(S, T) = |f|*



Upper bound on the flow

Lemma 26.5

The value of any flow *f* in a flow network *G* is bounded above by the capacity of any cut of G.



Max Flow-Min Cut Theorem

If *f* is a flow in a flow network *G=(V, E)* with source *s* and sink *t*, then the following conditions are equivalent:

1. *f* is a maximum flow in *G*
2. The residual network G_f contains no augmenting paths
3. $|f| = c(S, T)$ for some cut *(S, T)* in *G*

Prove that: IF *f* is a maximum flow in *G* THEN the residual network G_f contains no augmenting paths. *Which proof is correct?*

A. Suppose there is an augmenting path in G_f. Then the flow f can be augmented by the flow along the augmenting path f_p and we know that $|f \uparrow f_p| = |f| + |f_p| > |f|$. This is a contradiction. Therefore, there cannot be an augmenting path in G_f.

B. Suppose the residual network G_f contains no augmenting paths. Then we know that *f* is a maximum flow because it cannot be augmented any further.

C. Suppose *f* is a maximum flow in G. Since it is maximum, it cannot be augmented any further so there cannot be an augmenting path in G_f.

D. All of the above proofs will work.

E. None of the above proofs will work

Prove that: IF the residual network G_f contains no augmenting paths THEN $|f| = c(S, T)$ for some cut (S, T) in G

1. G_f has no path from s to t.

2. Let S be a set of vertices reachable from s in G_f. Let $T = V - S$.

3. The partition (S, T) is a cut, $s \in S$ and $t \in T$.

4. Consider a pair of vertices $x \in S$ and $y \in T$.
   - If (x, y) is in E, then $f(x, y) = c(x, y)$ (otherwise (x, y) would be in E_f and y would be reachable from x and thus reachable from s in G_f).
   - If (y, x) is in E, then $f(y, x) = 0$ (otherwise $c_f(x, y) = f(y, x) > 0$ and y would be reachable from x and thus reachable from s in G_f).
   - If neither (x, y), nor (y, x) are in E, then $f(x, y) = f(y, x) = 0$.

$f(S,T) = \sum_{x\in S}\sum_{y\in T} f(x,y) - \sum_{x\in S}\sum_{y\in T} f(y,x) = \sum_{x\in S}\sum_{y\in T} c(x,y) - 0 = c(S,T)$

By earlier Lemma 26.4 we know that if (S, T) is any cut in G, then $f(S, T) = |f|$.

Therefore $|f| = f(S, T) = c(S, T)$.

Prove that: IF $|f| = c(S, T)$ for some cut (S, T) in G THEN f is a maximum flow in G

Since $|f| = c(S, T)$ and by earlier lemma, we know that $|f| \le c(S, T)$ for all cuts (S, T), we know that the value of $|f|$ cannot be any larger than it already is (since it reached its upper bound). Therefore f is a maximum flow function.

The Basic Ford-Fulkerson Algorithm

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FORD-FULKERSON(G, s, t)
1. for each edge (u, v) in G.E
2.     (u, v).f = 0
3. while there is a path from s to t in G_f
4.     c_f(p) = min{c_f(u,v): (u, v) is in p}
5.     for each edge (u, v) in p
6.         if (u, v) in G.E
7.             (u, v).f = (u, v).f + c_f(p)
8.         else
9.             (v, u).f = (v, u).f – c_f(p)
```

Which of the following is correct?

A. If edge weights are integers, each iteration of the while loop (lines 3 – 9) increases the flow on G by at least 1.

B. If edge weights are integers, and the maximum flow is |f*|, then the algorithm executes in O(E|f*|).

C. The search for augmenting path from s to t (on line 3) can be done in linear time (liner in terms of |V| and |E|).

D. All of the above